ELLIPSE & HYPERBOLA

TANGENTS & NORMAL

EQUATION OF AN ELLIPSE :- An ellipse is the locus of a point which moves in a plane such that its distance from a fixed point bears a constant ratio to its distance from a fixed line and this ratio is less than unity. OR

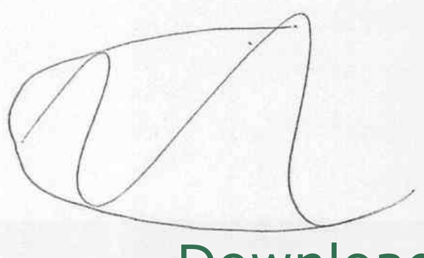
An ellipse is the set of points in a plane, the sum of whose distance from two fixed points is constant. Alternatively, an ellipse is the set of all points in the plane whose distances from a fixed point in the plane bears a constant ratio, less than, to their distance from a fixed line in the plane. The fixed point is called focus, the fixed line a directrix and the constant ratio (e) the eccentricity of the ellipse.

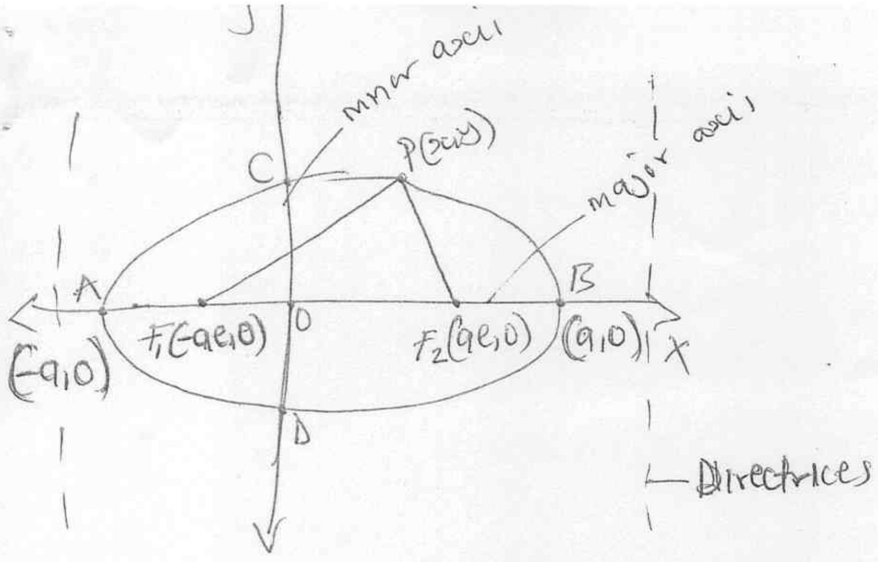
we have two standard form of the ellipse, i.e.

(i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ & (ii) $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$

in both cases $a > b$ & $b^2 = a^2(1-e^2)$, $e < 1$

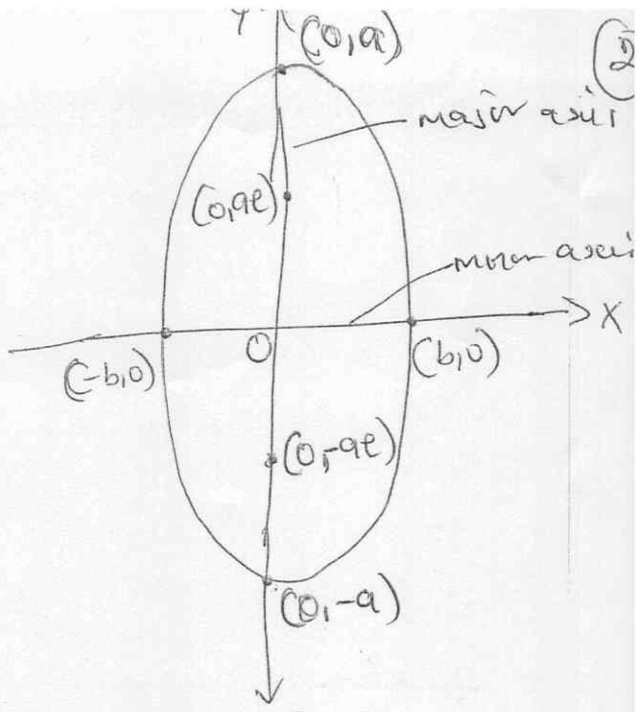
in (i) major axis is along x-axis & minor along y-axis & in (ii) major axis is along y-axis & minor along x-axis as shown below





$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

(a)



$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

(b)

FACTS ABOUT ELLIPSE

Form of the ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a>b$	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, $a>b$
equation of major axis	$y=0$	$x=0$
length of major axis	$2a$	$2a$
equation of minor axis	$x=0$	$y=0$
length of minor axis	$2b$	$2b$
directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{a}{e}$
equation of latus rectum	$x=\pm ae$	$y=\pm ae$
length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
centre	$(0,0)$	$(0,0)$
Foci	$(\pm ae,0)$	$(0,\pm ae)$

FOCAL DISTANCE :- The focal distance of a point (x, y) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$a - e|x|$ from the nearer focus

$a + e|x|$ " " farther focus.

Sum of the focal distances of any point on an ellipse is constant & equal to the length of the major axis.

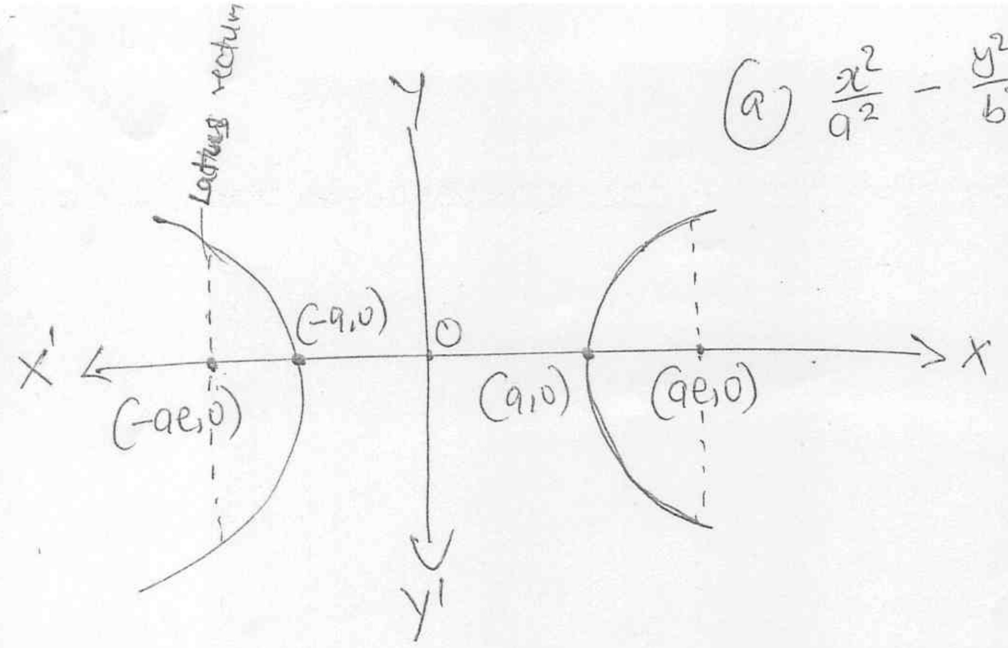
HYPERBOLA :- is the set of all points in a plane, the difference of whose distance from two fixed points is constant. Alternatively, a hyperbola is the set of all points in a plane whose distances from a fixed point in the plane bears a constant ratio, greater than one (1), to their distances from a fixed line in the plane. The fixed point is called a focus, the fixed line a directrix & the constant ratio denoted by e, the eccentricity of the hyperbola.

We have two standard forms of the hyperbola, i.e.,

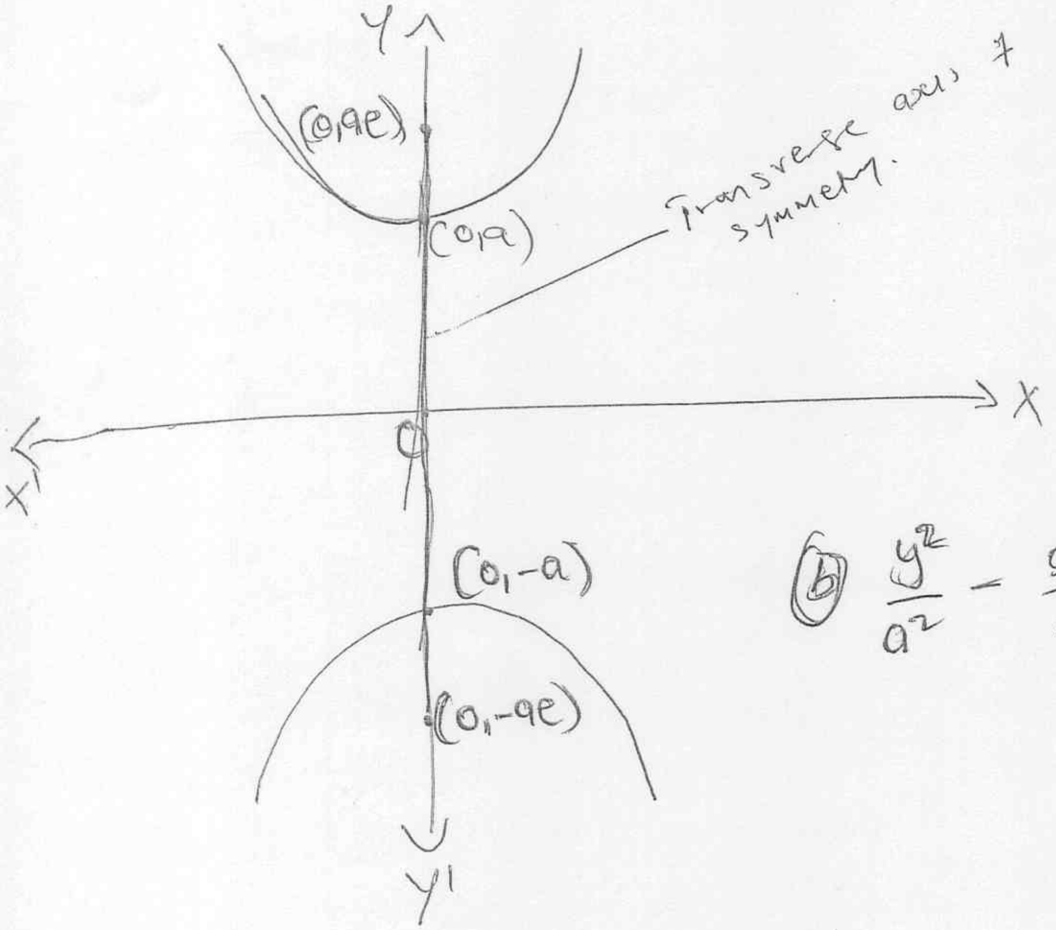
(i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & (ii) $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Here $b^2 = a^2(e^2 - 1)$, $e > 1$

In (i) transverse axis is along x-axis & conjugate axis along y-axis while as in (ii) transverse axis is along y-axis & conjugate axis along x-axis.



(a) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



(b) $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

FACTS ABOUT Hyperbola.

Forms of the hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Equation of transverse axis	$y = 0$	$x = 0$
Equation of conjugate axis	$x = 0$	$y = 0$
Length of transverse axis	$2a$	$2a$

Foci	$(\pm ae, 0)$	$(0, \pm ae)$
Equation of latus rectum	$x = \pm ae$	$y = \pm ae$.
Length of latus rectum	$\frac{2b^2}{a}$.	$\frac{2b^2}{a}$.
Centre	$(0,0)$	$(0,0)$.

FOCAL DISTANCE :- The focal distance of any point (x, y) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$e|x| - a$ from the nearer focus

$e|x| + a$ from " farther focus

Differences of the focal distances of any point on a hyperbola is constant and equal to the length of the transverse axis.

Ex ①
Given the Ellipse with equation $9x^2 + 25y^2 = 225$, find the major, minor axes, eccentricity, foci & vertices.

Soln.
Putting the given eqn in standard form, we have [$\because \div$ bothside by 225]

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

This shows that $a = 5$, $b = 3$

Hence $b^2 = a^2(1-e^2) \Rightarrow 9 = 25(1-e^2)$, $\Rightarrow e = \frac{4}{5}$.

Since the denominator of x^2 is larger, the major axis is along (6) x-axis, minor axis along y-axis,

foci are $(\pm ae, 0) \Rightarrow \pm\left(5 \times \frac{4}{5}\right) = \pm 4$

$\Rightarrow (4,0)$ & $(-4,0)$.

and vertices $\Rightarrow (-a,0)$ & $(a,0) \Rightarrow (-5,0)$ & $(5,0)$

EX 2.

Find the equation of the ellipse with foci at $(\pm 5, 0)$ & $x = \frac{6}{5}$ as one of the directrices.

Sol:.

we have $ae = 5$ & $\frac{a}{e} = \frac{6}{5} \Rightarrow a^2 = 36$ or $a = 6$.

erefor, $e = \frac{5}{6}$, But $b^2 = a^2(1-e^2) \Rightarrow b^2 = a^2 - a^2e^2$.

now $b = a\sqrt{1-e^2} = 6\sqrt{1 - \frac{25}{36}} = \sqrt{11}$. ; the eq of the

Ellipse is $\Rightarrow$

$$\frac{x^2}{36} + \frac{y^2}{11} = 1$$

EX.3

~~The equation of the hyperbola can be written as $\frac{x^2}{16} - \frac{y^2}{9} = 1$~~

for the hyperbola $9x^2 - 16y^2 = 144$, find the vertices, foci & eccentricity.

Sol:.

The eq of the hyperbola can be written as $\frac{x^2}{16} - \frac{y^2}{9} = 1$

$\Rightarrow a = 4$, $b = 3$ & $9 = 16(e^2-1)$

$\Rightarrow 9 = 16e^2 - 16 \Rightarrow 9 + 16 = 16e^2 \Rightarrow e^2 = \frac{9}{16} + \frac{16}{16}$

$e^2 = \frac{9}{16} + 1 = \frac{25}{16}$

$\Rightarrow e = \frac{5}{4}$

$\Rightarrow$ vertices are $(\pm a, 0)$

$\Rightarrow (4, 0)$ & $(-4, 0)$ &

foci are $(\pm ae, 0)$ (but $ae \Rightarrow \frac{5}{4} * 4 = 5$

$\Rightarrow (\pm 5, 0)$

Ex. 4
Find the equation of the hyperbola with vertices at $(0, \pm 6)$ & $e = \frac{5}{3}$.
Find it's foci.

Sol. Since the vertices are on the y-axes we have
$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. as $\bar{e}$ equation.

~~but $(0, \pm ae) = (0, \pm 6) \Rightarrow ae = 6$ & $e = \frac{5}{3}$~~

~~$a\left(\frac{5}{3}\right) = 6 \Rightarrow 5a = 18 \Rightarrow a = \frac{18}{5}$~~

as given the vertices

$(0, \pm 6) = (0, \pm a)$, $\Rightarrow a = 6$. $\left[6 * \frac{5}{3} = ae, \Rightarrow \frac{30}{3} = 10\right.$

but $b^2 = a^2(e^2 - 1) \Rightarrow b^2 = 36\left[\frac{25}{9} - 1\right] = 64$.

the required equation of the hyperbola is

$\frac{y^2}{36} - \frac{x^2}{64} = 1$, it's foci is $(0, \pm ae) = (0, \pm 10)$

Ex

find the equation of the ellipse which passes through the point $(-3,1)$ & has eccentricity $\frac{\sqrt{2}}{5}$, with x-axis as it's major axis & centre at the origin.

soln.

let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be eqn of ~~the~~ ellipse passing through the point $(-3,1)$.

$\Rightarrow \frac{9}{a^2}+\frac{1}{b^2}=1$, $\Rightarrow 9b^2+a^2=a^2b^2$, But $b^2=a^2(1-e^2)$

$\Rightarrow$ ~~$9a^2(1-e^2)+a^2=a^2b^2$~~.

$\Rightarrow 9a^2(1-e^2)+a^2=a^2 * a^2(1-e^2)$.

$\Rightarrow 9a^2-9a^2e^2+a^2=a^4(1-e^2)$.

$\Rightarrow a^2(9-9e^2+1)=a^4(1-e^2)$

$\Rightarrow 9-9e^2+1=a^2(1-e^2)$, But $e=\frac{\sqrt{2}}{5}$, $e^2=\frac{2}{25}$.

$\Rightarrow 9-9(2/25)+1=a^2\left(1-\frac{2}{25}\right)$

$\Rightarrow 9-\frac{18}{25}+1=a^2\left(\frac{23}{25}\right)$

$\Rightarrow \frac{232}{25}=a^2\left(\frac{23}{25}\right) \Rightarrow \frac{232*25}{25*23}=a^2$, $\Rightarrow a^2=\frac{5800}{575}$

$$\frac{9b^2+a^2}{a^2b^2}=1, \Rightarrow 9b^2+a^2=a^2b^2$$

But $b^2=a^2(1-e^2)$ & $e=\frac{\sqrt{2}}{5} \Rightarrow e^2=\frac{2}{25}$

$$\Rightarrow 9a^2(1-e^2)+a^2=a^2 * a^2(1-e^2).$$

$$\Rightarrow 9a^2\left[1-\frac{2}{25}\right]+a^2=a^4\left[1-\frac{2}{25}\right].$$

$$\Rightarrow \frac{207a^2}{25}+a^2=a^4\left(\frac{23}{25}\right).$$

$$\Rightarrow a^2\left[\frac{207}{25}+1\right]=a^4\left(\frac{23}{25}\right), \Rightarrow \frac{207}{25}+1=\frac{23a^2}{25}.$$

$$\Rightarrow \frac{232}{25}=\frac{23a^2}{25} \Rightarrow \frac{232*\cancel{25}}{\cancel{25}*23}=a^2, \Rightarrow a^2=\frac{232}{23}.$$

$$\Rightarrow b^2=a^2(1-e^2) \Rightarrow b^2=\frac{232}{23}\left[1-\frac{2}{25}\right] \Rightarrow b^2=\frac{232}{25}.$$

Hence, the required equation of the ellipse is

$$\frac{x^2}{\frac{232}{23}}+\frac{y^2}{\frac{232}{25}}=1$$

or $$=\frac{23x^2}{232}+\frac{25y^2}{232}=1.$$

or $$23x^2+25y^2=232.$$

Ex

Find the equation of the hyperbola whose vertices are $(\pm 6, 0)$ & one of the directrices is $x = 4$.

Soln.

As the vertices are on the x-axis & their middle point is the origin, the equation is of the type

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

we know that $b^2 = a^2(e^2 - 1)$, vertices are $(\pm a, 0)$ & directrices are given by $x = \pm \frac{a}{e}$.

Thus $a = 6$ & $\frac{a}{e} = 4 \Rightarrow \frac{6}{e} = 4, \Rightarrow 4e = 6$

$\Rightarrow e = \frac{6}{4} \Rightarrow e = \frac{3}{2}$.

$\Rightarrow b^2 = 36\left[\frac{9}{4} - 1\right] = 45$.

The required equation of the hyperbola is

$$\frac{x^2}{36} - \frac{y^2}{45} = 1$$

//.

The $\perp$ distance of a given point from a line

Consider the diagram below, DE is a line with eqn. $Ax + By + C = 0$ and slope $= -\frac{A}{B}$.

Let a point $P(m, n)$ be a point $\perp$ to DE at Q.

We wish to find the $\perp$ distance from P to line DE.

i.e distance PQ.

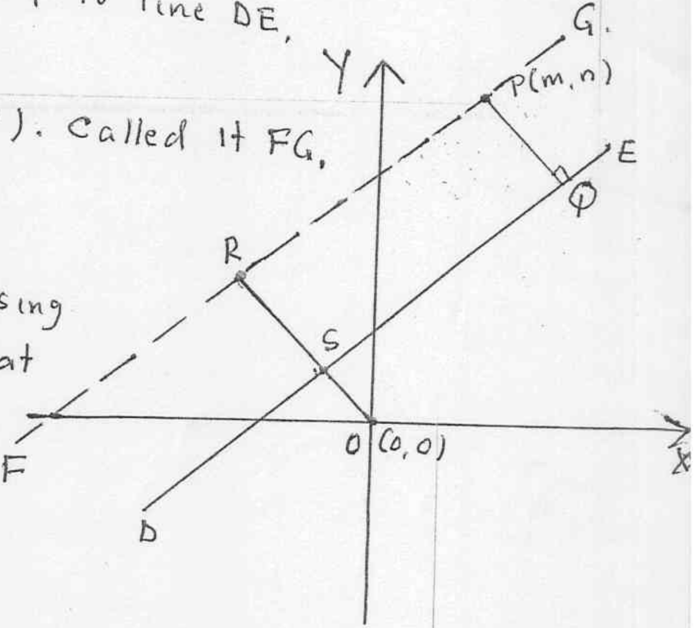
Construct a line $\cancel{\parallel}$ to DE thru. (m, n). Called it FG.

Its slope is $\cancel{\frac{B}{A}}$ $-\frac{A}{B}$.

Construct another line $\cancel{\parallel}$ to PQ passing thru. the origin. Intersecting DE and FG at S and R respectively.

RS slope is $\frac{B}{A}$ because it is $\perp$ to DE.

$RS = PQ$.



Since FG passes thru. (m, n) and has slope $-\frac{A}{B}$ its equation is

$$y - n = -\frac{A}{B}(x - m) \quad \text{or} \quad y = \frac{-Ax + Am + Bn}{B}$$

Line RS has equation

$$y - 0 = \frac{B}{A}(x - 0)$$

$$y = \frac{B}{A}x$$

Line FG intersect with line RS, so the values of x and y are equal at the point of intersection.

$$\frac{B}{A}x = \frac{-Ax + Am + Bn}{B}$$

$$\frac{B}{A}x + \frac{Ax}{B} = \frac{Am + Bn}{B}$$

$$x\left(\frac{B}{A} + \frac{A}{B}\right) = \frac{Am + Bn}{B}$$

$$\Rightarrow x\frac{(B^2+A^2)}{AB} = \frac{Am+Bn}{B}$$

$$x = \frac{Am+Bn}{\cancel{B}}\cdot\frac{A\cancel{B}}{A^2+B^2} = \frac{A(Am+Bn)}{A^2+B^2}$$

Subsituting $x = \frac{A(Am+Bn)}{A^2+B^2}$ in $y = \frac{B}{A}x$ gives,

$$y = \frac{B}{A}\left\{\frac{A(Am+Bn)}{A^2+B^2}\right\} = \frac{B(Am+Bn)}{A^2+B^2}$$

So point R has coordinates $\left\{\frac{A(Am+Bn)}{A^2+B^2}, \frac{B(Am+Bn)}{A^2+B^2}\right\}$

Point S is the intersection of RS with DE.

RS has eqn $y = \frac{B}{A}x$ while DE has eqn. $Ax+By+C=0$

$$\Rightarrow y = \frac{-Ax-C}{B}$$

At the point of intersection the values of x and y are the same so

$$\frac{B}{A}x = \frac{-Ax-C}{B} \Rightarrow \frac{B}{A}x + \frac{Ax}{B} = -\frac{C}{B}$$

$$x\left(\frac{B}{A}+\frac{A}{B}\right) = -\frac{C}{B}$$

$$x\left(\frac{B^2+A^2}{AB}\right) = -\frac{C}{B}$$

$$x = -\frac{C}{\cancel{B}}\cdot\frac{A\cancel{B}}{A^2+B^2} = \frac{-AC}{A^2+B^2}$$

Subsituting $x = \frac{-AC}{A^2+B^2}$ in $y = \frac{B}{A}x$,

$$y = \frac{B}{\cancel{A}}\cdot -\frac{\cancel{A}C}{A^2+B^2} = -\frac{BC}{A^2+B^2}.$$

S has coordinates $\left(\frac{-AC}{A^2+B^2}, \frac{-BC}{A^2+B^2}\right)$

The distance RS, using distant formula,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{is}$$

$$d = \sqrt{\left(\frac{-AC}{A^2+B^2} - \frac{A(Am+Bn)}{A^2+B^2}\right)^2 + \left(\frac{-BC}{A^2+B^2} - \frac{B(Am+Bn)}{A^2+B^2}\right)^2}$$

$$= \sqrt{\frac{\{-A(Am+Bn+C)\}^2 + \{-B(Am+Bn+C)\}^2}{(A^2+B^2)^2}}$$

$$= \sqrt{\frac{A^2(Am+Bn+C)^2 + B^2(Am+Bn+C)^2}{(A^2+B^2)^2}}$$

$$= \sqrt{\frac{(Am+Bn+C)^2\cancel{(A^2+B^2)}}{(A^2+B^2)^{\cancel{2}}}} = \sqrt{\frac{(Am+Bn+C)^2}{A^2+B^2}}$$

$$d = \frac{|Am+Bn+C|}{\sqrt{A^2+B^2}}$$

Example: Find the equation of the line CP passing through the point C(1, 2) and making an angle 90° with the line $x - \sqrt{3}y + 4 = 0$, Find also the ⊥ distance of CP from the origin.

Soln:

The equation $x - \sqrt{3}y + 4 = 0$ can be rewritten as

$$-\sqrt{3}y = -x - 4$$

$$y = +\frac{x}{\sqrt{3}} + \frac{4}{\sqrt{3}}$$

$$y = \frac{x}{\sqrt{3}} + \frac{4}{\sqrt{3}}.$$

$\Longrightarrow$ the slope of the line $(x - \sqrt{3}y + 4 = 0)$ is $\frac{1}{\sqrt{3}} = m_1$

Since the line CP is ⊥ to line $x - \sqrt{3}y + 4 = 0$, the product of their slope $= -1$. Let the slope the line CP passing be m_2.

Then $m_1 m_2 = -1$

$$\Rightarrow \frac{1}{\sqrt{3}} m_2 = -1$$

$$m_2 = -\sqrt{3}$$

So the equation of line CP passing thru. (1, 2) is

$$y - 2 = -\sqrt{3}(x - 1)$$

$$y = -\sqrt{3}x + \sqrt{3} + 2$$

$$y = -\sqrt{3}x + 2 + \sqrt{3}$$

$\Rightarrow +\sqrt{3}x + y - 2 - \sqrt{3} = 0$ is the required equation CP.

The $\perp$ distance $p(h, K)$ from the line $Ax + By + C = 0$ is given by

$$p = \frac{Ah + BK + C}{(A^2 + B^2)^{1/2}}, \quad \text{but } (h, K) = (0, 0)$$

$A = \sqrt{3}$, $B = 1$, $C = -(2 + \sqrt{3})$

So,

$$p = \frac{\sqrt{3}\cdot 0 + 1\cdot 0 + (2 + \sqrt{3})}{[(\sqrt{3})^2 + 1^2]^{1/2}} = \frac{-(2 + \sqrt{3})}{(3 + 1)^{1/2}}$$

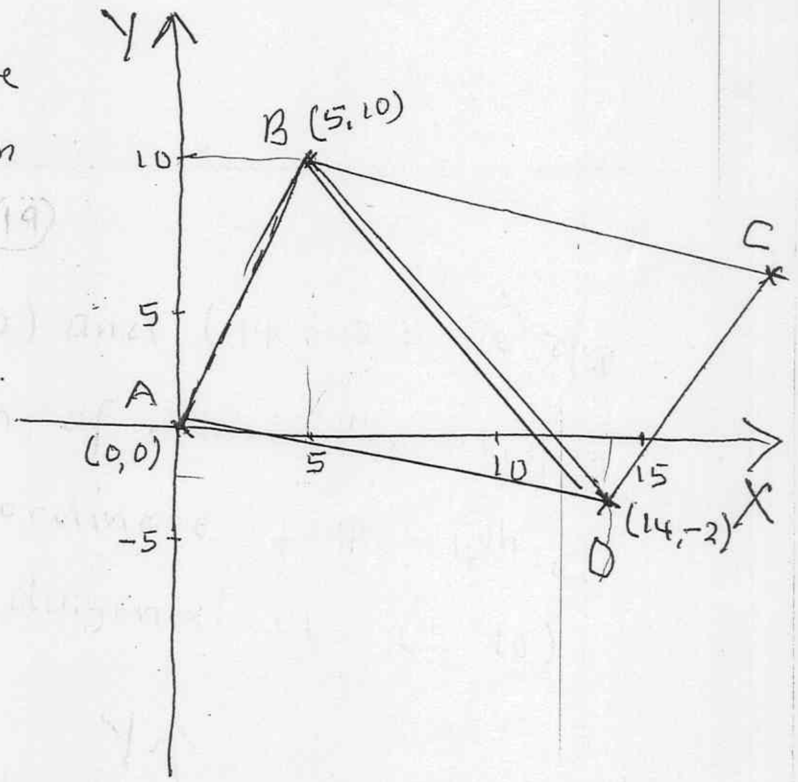
$$= \frac{-(2 + \sqrt{3})}{4^{1/2}} = \frac{-(2 + \sqrt{3})}{2}.$$

The required distance $p = -\left(1 + \frac{\sqrt{3}}{2}\right)$

$$= +1 + \frac{\sqrt{3}}{2}$$

Example: The points (5, 10) and (14, −2) are opposite corners of a parallelogram, of which the origin is a third corner. Find the coordinate of the 4th corner and the equation of the diagonal via (5, 10).

Soln:



Since a //logram has opposite sides are equal and ~~//~~, then

slope(AB) = slope(CD)

$$\text{Slope}(AB) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10-0}{5-0}$$

$$= 2$$

$$\text{Slope}(CD) = \frac{y+2}{x-14} = 2$$

$$y + 2 = 2(x - 14)$$

$$y - 2x = -2 - 28$$

$$y - 2x = -30 \quad \ldots (1)$$

$$\text{Slope}(AD) = \text{Slope}(BC)$$

$$\text{Slope}(AD) = \frac{-2-0}{14-0} = -\frac{1}{7}.$$

$$\text{Slope}(BC) = \frac{y-10}{x-5} = -\frac{1}{7}$$

$$7y - 70 = -x + 5$$

$$\Rightarrow \quad 7y + x = 75 \quad \ldots (2)$$

Solving (1) and (2) Simultaneously.

$\cancel{7y = 14x}$

$$y - 2x = -30$$

$$14y + 2x = 150$$

$$15y = 120$$

$$y = 8 //$$

put $y = 8$ in (1)

$$8 - 2x = -30$$

$$-2x = -30 - 8$$

$$x = 19 //$$

The corner C coordinates are $(19, 8)$ //

The Equation of the diagonal from point $(5, 10)$ to $(14, -2)$ can be obtained using two points formula,

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y - 10}{-2 - 10} = \frac{x - 5}{14 - 5}$$

$$\frac{y - 10}{-12} = \frac{x - 5}{9}$$

$$9y - 90 = -12x + 60$$

$$9y + 12x = 150$$

$3y + 4x = 50$ is the equation of the diag-