# MTH 201: Mathematical Method 1.

Real-valued functions of a real variable. Review of differenciation and Integration and their applications. Mean value theorem. Taylor's Series. Real-valued functions of two or ~~more~~ three variables, partial derivatives, chain rule, extrema, Langranges Multipliers. Increments, differentiable and linear approximations. Evaluation of line Integrals. Multiple Integrals.

FUNCTIONS: A variable $y$ is a function of another variable $x$ (written as $y = F(x)$) if the value of $y$ is determined by the value of $x$, that is $y$ is dependent on $x$. The letter $x$ is called the Independent variable and the letter $y$ is called the dependent variable. A Function $F$ is a rule that assigns each element $x$ in a set $A$ exactly one element, called $f(x)$ in a set $B$. Here we consider functions for which sets $A$ and $B$ are sets of real numbers.

DOMAIN AND RANGE: The set $A$ as defined above is called the domain of the function, that is, the set of all numbers that are assigned to the Independent variable. The set $B$ is called the co-domain of the function. $F: A \to B$ is ~~called~~ represents a function that establishes a correspondance between sets $A$ and $B$. Let $x \in A$ and $f(x) \in B$. The element $f(x)$ in $B$ is called the image of $x \in A$. The subset of the co-domain, which is a collection of all the images of the elements of the domain is called the range. The range is also the set of all possible values of the dependent variable $f(x)$

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DIFFERENTIATION . .

If $y = f(x)$ then

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - F(x)}{\Delta x}$$

If There exist a limit of the ratio $\frac{\Delta y}{\Delta x}$ as $(\Delta x \to 0)$ $\Delta x$ approches Zero, Then $\frac{\Delta y}{\Delta x}$ approches a limiting value an This limit is called the derivative of the function $y = f(x)$ at the point $x$ and is denoted by $dy/dx$ or $f'(x)$ or $y'(x)$. Thus;

$$\frac{dy}{dx} = f'(x) = y'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

In general;

1) If $y = ax^n$; $\frac{dy}{dx} = nax^{n-1}$

e.g if $y = 5x^4$; $\frac{dy}{dx} = 4 \times 5x^{4-1} = 20x^3$

2) $y = C$, $\frac{dy}{dx} = 0$

3) $y = 5x$; $\frac{dy}{dx} = 5$

4) $y = \sqrt{x}$; $\frac{dy}{dx} = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}$

5) $y = \sin x$; $\frac{dy}{dx} = \cos x$

6) $y = \cos x$; $\frac{dy}{dx} = -\sin x$

## Identities

① $\sec x = \dfrac{1}{\cos x}$

② $\operatorname{cosec} x = \dfrac{1}{\sin x}$

③ $\cot x = \dfrac{\cos x}{\sin x}$

④ $\tan x = \dfrac{\sin x}{\cos x}$

⑤ $\sin^2 x + \cos^2 x = 1$

⑥ $1 + \tan^2 x = \sec^2 x$

⑦ $1 + \cot^2 x = \operatorname{cosec}^2 x$

### CHAIN RULE (FUNCTION OF FUNCTION)

If $y$ is a function of $u$ where $u$ is some function of $x$ Then,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \qquad [\, y = f(u); \quad u = f(x) \,]$$

Example: Differenciate the fallowing w.r.t $x$

① $\dfrac{3}{\sqrt{3x^2+5x+1}}$ ② $(5x^2+x+3)^4$ ③ $(\cos x)^2$

④ $\sec^2 x$ ⑤ $\sqrt[3]{2x^2+x^3}$ ⑥ $\sin(2x+3)$

⑦ $\tan^2(3x-4)$ ⑧ $\left(\dfrac{\cos x}{1+\sin x}\right)^2$ ⑨ $\operatorname{cosec} x$ ⑩ $\sec x$

Soln

① Let $y = \dfrac{3}{\sqrt{3x^2+5x+1}}$

$$y = \frac{3}{(3x^2+5x+1)^{\frac{1}{2}}} = 3(3x^2+5x+1)^{-\frac{1}{2}}$$

Let $u = 3x^2+5x+1$ Then $y = 3u^{-\frac{1}{2}}$; $\dfrac{dy}{du} = \dfrac{-3}{2\sqrt{u}}$

$$\frac{du}{dx} = 6x+5$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{-3}{2\sqrt{u}} \times (6x+5) = -3(6x+5)$$

② let $y = (5x^2 + x + 3)^4$

$u = 5x^2 + x + 3$ ; $\quad y = u^4$

$\dfrac{du}{dx} = 10x + 1 \qquad \dfrac{dy}{du} = 4u^3$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 4u^3 \times (10x+1)$$
$$= 4(10x+1)(5x^2+x+3)^3$$

③ $y = (\cos x)^2$

$u = \cos x \qquad y = u^2$

$\dfrac{du}{dx} = -\sin x \qquad \dfrac{dy}{du} = 2u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2u \times (-\sin x)$$
$$= -2 \sin x \cos x$$

④ $y = \sec^2 x = \left(\dfrac{1}{\cos x}\right)^2 = (\cos x)^{-2}$

$u = \cos x \qquad y = u^{-2}$

$\dfrac{du}{dx} = -\sin x \qquad \dfrac{dy}{du} = -2u^{-3}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\sin x \times (-2u^{-3}) = \frac{-2\sin x}{\cos x} \cdot \frac{-1}{\cos^2 x}$$
$$= 2 \tan x \sec^2 x$$

5) Exercise.

6) Exercise

## PRODUCT RULE.

Let $u, v$ and $w$ be differentiable functions of $x$

if $y = uv$ then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

if $y = uvw$ then

$$\frac{dy}{dx} = uv\frac{dw}{dx} + w\frac{d}{dx}(uv)$$
$$= uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

Example, Differentiate the following functions;

(1) $y = x^6 \cos x$ (2) $y = \sqrt{x}(x^4+3)$ (3) $(x^2+4)^2(2x^3-1)^3$

(4) $x^2 \sin 2x$ (5) $\cos^7 4x$ (6) $6x^2 \sin x \cos x$

Soln

(1) $y = x^6 \cos x$

Let $u = x^6$ and $v = \cos x$

Then $\frac{du}{dx} = 6x^5$; and $\frac{dv}{dx} = -\sin x$

$\therefore \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= x^6 \cdot (-\sin x) + \cos x \cdot 6x^5$

$= -x^6 \sin x + 6x^5 \cos x$

(2) $y = \sqrt{x}(x^4+3)$

Let $u = \sqrt{x}$; $v = x^4 + 3$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ ; $\frac{dv}{dx} = 4x^3$

Then, $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = \sqrt{x}(4x^3) + (x^4+3)\cdot\frac{1}{2\sqrt{x}}$

$= \sqrt{x}(4x^3) + \frac{x^4+3}{2\sqrt{x}}$

$= 4x^{7/2} + \frac{x^4+3}{2\sqrt{x}}$.

Exercises, (3), (4) and (5)

(6) $y = 6x^2 \sin x \cos x$

$\frac{dy}{dx} = 6x^2 \sin x \frac{d}{dx}(\cos x) + 6x^2 \cos x \frac{d}{dx}(\sin x) + \sin x \cos x \frac{d}{dx}(6x^2)$

$= -6x^2 \sin^2 x + 6x^2 \cos^2 x + 12x \sin x \cos x$

$= 6x^2(\cos^2 x - \sin^2 x) + 12x \sin x \cos x$

$= 6x^2(\cos^2 x - \ldots$

$= 6x^2\cos 2x + 6x\sin 2x = 6x(\sin 2x + x\cos 2x)$.

## QUOTIENT RULE

Let $u$ and $v$ be differentiable functions of $x$ if $y = u/v$ then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

**Examples;** (1) $\frac{4x^3 - 7x^2 - 2}{3x+5}$ (2) $\tan x$ (3) $\cot x$ (4) $\frac{\sin x}{x^2+\cos x}$ (5) $\frac{x+\sin x}{1+\cos x}$ (6) $\frac{x\sin x}{\cos x+\sin x}$ (7) $\frac{x(x+1)}{(x+2)(x+3)}$ (8) $u = \frac{\theta\cos\theta}{\theta+3}$ (9) $u = \frac{\theta\cos\theta}{(\theta+1)\sin\theta}$ (10) $\frac{1}{1+\cos x}$

## Soln

(1) let $y = \tan x = \frac{\sin x}{\cos x}$

$u = \sin x$, $\quad v = \cos x$

$\frac{du}{dx} = \cos x \qquad \frac{dv}{dx} = -\sin x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{(\cos x)^2} = \frac{\cos^2 x - \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

(8) $u = \dfrac{\theta \cos\theta}{\theta + 3}$

$$\frac{du}{d\theta} = \frac{(\theta+3)\frac{d}{d\theta}(\theta\cos\theta) - \theta\cos\theta\frac{d}{d\theta}(\theta+3)}{(\theta+3)^2}$$

$$= \frac{(\theta+3)(\cos\theta - \theta\sin\theta) - \theta\cos\theta \cdot 1}{(\theta+3)^2}$$

$$= \frac{3\cos\theta - (\theta+3)\theta\sin\theta}{(\theta+3)^2}$$

(9) $u = \dfrac{\theta\cos\theta}{(\theta+1)\sin\theta}$

$$\frac{du}{d\theta} = \frac{(\theta+1)\sin\theta\frac{d}{d\theta}(\theta\cos\theta) - \theta\cos\theta\frac{d}{d\theta}[(\theta+1)\sin\theta]}{(\theta+1)^2\sin^2\theta}$$

$$= \frac{(\theta+1)\sin\theta(\cos\theta - \theta\sin\theta) - \theta\cos\theta[(\theta+1)\cos\theta + \sin\theta]}{(\theta+1)^2\sin^2\theta}$$

$$= \frac{\sin\theta\cos\theta - \theta(\theta+1)}{(\theta+1)^2\sin^2\theta}$$

## Derivative of Exponential Function

Note $\log_e^e = 1$ ; $\log_e^a = \ln e$

If $y = e^{f(x)}$, then $\dfrac{dy}{dx} = f'(x)e^{f(x)}$

If $y = a^{f(x)}$; Then, $\dfrac{dy}{dx} = f'(x)\, a^{f(x)} \ln a$

Examples; (1) $e^x$ (2) $a^x$ (3) $e^{x^2/2 - 6x + 3}$ (4) $4^{-x^2+3}$

(5) $a^{3x^2}$

③ $y = e^{x^2/2 - 6x + 3}$

$f(x) = x^2/2 - 6x + 3; \quad f'(x) = x - 6$

$$\frac{dy}{dx} = (x-6)\, e^{x^2/2 - 6x + 3}$$

④ $y = 4^{-x^2+3}$

$f(x) = -x^2 + 3; \quad f'(x) = -2x$

$$\frac{dy}{dx} = -2x\, 4^{-x^2+3} \ln 4$$

Exercises (1), (2) and (5).

## Derivative of Logarithmic Functions

If $y = \log_e f(x)$ then $\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$

If $y = \log_a f(x)$ then $\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x) \ln a}$

Examples; Differentiate the following

① $y = \ln(3x^2 - 6x + 8)$ ② $\log_a (3x^2 - 5)$

Soln

① $y = \ln(3x^2 - 6x + 8)$

$f(x) = 3x^2 - 6x + 8; \quad f'(x) = 6x - 6$

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)} = \frac{6x - 6}{3x^2 - 6x + 8}$$

② $y = \log_a (3x^2 - 5)$

$f(x) = 3x^2 - 5; \quad f'(x) = 6x$

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)\ln a} = \frac{6x}{(3x^2-5)\ln a}$$

Examples;

$$y = xe^x + \frac{\sin x}{\ln x}$$

$$\frac{dy}{dx} = xe^x + e^x + \frac{\ln x \cdot \cos x - \frac{\sin x}{x}}{(\ln x)^2}$$

$$= xe^x + e^x + \frac{x \ln x \cos x - \sin x}{x(\ln x)^2}$$

## Implicit differentiation

Suppose, for example, that $y$ is defined as an implicit function of $x$ by the equation $x^2+y^2=4$ we differentiate each term w.r.t $x$ and obtain

$$2x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Examples; find $\frac{dy}{dx}$ if $x^2+y^2+\sin y = 3$

Soln

$$2x\frac{dx}{dx} + 2y\frac{dy}{dx} + \cos y\frac{dy}{dx} = 0$$

$$(2y + \cos y)\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y+\cos y}$$

Example; find $\frac{dy}{dx}$ if $x^3+y^3 = 6xy^4$

Soln

$$x^3+y^3 = 6xy^4$$

$$3x^2 \frac{dx}{dx} + 3y^2 \frac{dy}{dx} = 6x \cdot 4y^3 \frac{dy}{dx} + y^4 \cdot 6 \frac{dx}{dx}$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 24xy^3 \frac{dy}{dx} + 6y^4$$

$$(3y^2 - 24xy^3) \frac{dy}{dx} = 6y^4 - 3x^2$$

$$\frac{dy}{dx} = \frac{6y^4 - 3x^2}{3y^2 - 24xy^3} = \frac{2y^4 - x^2}{y^2 - 8xy^3}$$

**Exercise**: find $\frac{dy}{dx}$ if (1) $xy + x - 2y - 1 = 0$

(2) find $\frac{dy}{dx}$ and $\frac{\partial^2 y}{\partial x^2}$ if $x + y + \sin y = 3$

(3) find $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$ at $(1,1)$ on the curve $x^2 + y^2 = 2$

(4) find $\frac{dy}{dx}$ when (a) $x + y + \cos x + \cos y = 2$

(b) $xy + \sin y = 1$

(c) at $(1,1)$ if $x^2 + y^2 + xy = 3$.

## Inverse Functions.

**Example**; if $y = \cos^{-1} x$; find $\frac{dy}{dx}$

**Soln**

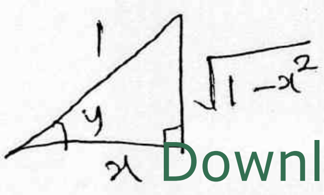
$$y = \cos^{-1} x \Rightarrow \cos y = x$$

differentiating implicitly gives

$$-\sin y \frac{dy}{dx} = \frac{dx}{dx}$$

$$-\sin y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y}$$

but $\cos y = \frac{x}{1}$




$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

Example; If $y = \tan^{-1} x$, find $\frac{dy}{dx}$

Soln

$y = \tan^{-1} x \Rightarrow \tan y = x$

differentiating implicitly gives;

$$\sec^2 y \frac{dy}{dx} = \frac{dx}{dx}$$

$$\sec^2 y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{x^2+1}$$

Example; find $\frac{du}{d\theta}$ if (i) $u = \arcsin 3\theta$
(ii) $u = \arctan(3\theta^3)$

Soln

(1) $u = \arcsin 3\theta = \arcsin v$; where $v = 3\theta$

$$\frac{du}{d\theta} = \frac{du}{dv} \cdot \frac{dv}{d\theta} \quad ; \quad \frac{dv}{d\theta} = 3$$

$u = \arcsin v = \sin^{-1} v$

$\Rightarrow v = \sin u$

$$\frac{du}{dv} = \frac{1}{\sqrt{1-v^2}}$$

$$\frac{du}{d\theta} = \frac{du}{dv} \cdot \frac{dv}{d\theta} = \frac{1}{\sqrt{1-v^2}} \times 3 = \frac{3}{\sqrt{1-v^2}}.$$

(2) $u = \arctan(3\theta^3) = \arctan v$; where $v = 3\theta^3$

$$\frac{du}{d\theta} = \frac{du}{dv} \cdot \frac{dv}{d\theta} = \frac{1}{1+v^2} \times 9\theta^2 = \frac{9\theta^2}{1+v^2}.$$

Example; find $\frac{d}{dx}\left[\arcsin\left(\frac{1-x^2}{1+x^2}\right)\right]$

Soln

let $y = \arcsin\left(\frac{1-x^2}{1+x^2}\right) = \arcsin u$; $u = \frac{1-x^2}{1+x^2}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot \frac{\ldots}{(1+x)^2}$$

$$= \frac{1}{\sqrt{\left(1 - \frac{(1-x^2)^2}{(1+x^2)^2}\right)}} \times \frac{-4x}{(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{-4x}{(1+x^2)^2} \times \frac{(1+x^2)}{2x} = \frac{-2}{1+x^2}.$$

Exercises; find the derivatives of the following

(1) $\operatorname{arc}\cot x$ (2) $\operatorname{arc}\operatorname{cosec} x$ (3) $\operatorname{arc}\sin x^2$ (4) $\operatorname{arc}\tan x^2$.

(5) $\operatorname{arc}\sin \frac{x}{a}$ (6) $\operatorname{arc}\sin\left(\frac{1-x}{1+x}\right)$ (7) $\operatorname{arc}\tan\left(\frac{1+x}{1-x}\right)$

(8) $\operatorname{arc}\tan \frac{x}{a}$ (9) $\operatorname{arc}\cos 6x$ (10) $\operatorname{arc}\tan\left(\frac{1+\tan x}{1-\tan x}\right)$

(11) $\operatorname{arc}\sec x$ (12) $\operatorname{arc}\cot \frac{x}{a}$.

## HIGHER ORDER DERIVATIVES.

$\frac{dy}{dx} = f'(x) = y' \longrightarrow$ 1st derivative

$\frac{d^2y}{dx^2} = f''(x) = y'' \longrightarrow$ 2nd derivative

$\frac{d^3y}{dx^3} = f'''(x) = y''' \longrightarrow$ 3rd derivative.

$\vdots$

$\frac{d^ny}{dx^n} = f^n(x) = y^n \longrightarrow$ nth derivative.

Examples; if $y = xe^x + 3x^2$; find the 2nd and 3rd derivative.

Soln

$y = xe^x + 3x^2$

$y' = xe^x + e^x + 6x$

$y'' = xe^x + e^x + e^x + 6$

$= xe^x + 2e^x + 6$

$y''' = xe^x + e^x + 2e^x$

$= xe^x + 3e^x$

$= e^x(x+3)$.

Example; find the Second derivative if $y = \frac{\sin\theta}{1+\cos\theta}$

Soln

$$y = \frac{\sin\theta}{1+\cos\theta}$$

$$\frac{dy}{d\theta} = \frac{(1+\cos\theta)\cos\theta - \sin\theta(-\sin\theta)}{(1+\cos\theta)^2}$$

$$= \frac{\cos^2\theta + \cos\theta + \sin^2\theta}{(1+\cos\theta)^2} = \frac{1+\cos\theta}{(1+\cos\theta)^2} = \frac{1}{1+\cos\theta}$$

$$\frac{d^2y}{d\theta^2} = \frac{-(-\sin\theta)}{(1+\cos\theta)^2} = \frac{\sin\theta}{(1+\cos\theta)^2}$$

Example; if $y = \tan\theta$, Show that $\frac{d^2y}{d\theta^2} = 2y(1+y^2)$

Soln

$$y = \tan\theta$$

$$\frac{dy}{d\theta} = \sec^2\theta = \sec\theta \cdot \sec\theta$$

$$\frac{d^2y}{d\theta^2} = \sec\theta \cdot \sec\theta\tan\theta + \sec\theta \cdot \sec\theta\tan\theta$$

$$= 2\tan\theta\sec^2\theta$$

$$= 2\tan\theta(1+\tan^2\theta) = 2y(1+y^2)$$

where $\boxed{y = \tan\theta}$.

Exercises; find $\frac{d^2y}{dx^2}$ if ① $y = \cos^2 x$ ② $y = \frac{\cos x}{1-\sin x}$

③ $y = e^{\sin x}$ ④ if $y = \sec\theta$, Show that $\frac{d^2y}{d\theta^2} = y(2y^2-1)$

APPLICATION OF DIFFERENTIATION

(See material).

Defn: A power Series is a Series of the form:

$$f(x) = \sum_{n=0}^{n} C_n x^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$$

Taylor's Series; The expansion for $f(x)$ is given by

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(a)$$

Maclaurin's Series; This Series is given by

$$f(x) = f(0) + x f'(0) + x^2 \frac{f''(0)}{2!} + \cdots + x^n \frac{f^n(0)}{n!}$$

NOTE; Maclaurin's Series is Taylor's Series at $a=0$

Examples; find the Taylor's Series expansion of each of the ff functions at the given point of $a$.

i. $f(x) = e^x$ at $a = 10$

ii. $f(x) = \cos x$ at $a = 2\pi$

iii. $f(x) = \frac{2}{(1+x)^2}$ at $a = 1$

Soln

(1) $f(x) = e^x$ $\quad f(10) = e^{10}$

$f'(x) = e^x$ $\quad f'(10) = e^{10}$

$f''(x) = e^x$ $\quad f''(10) = e^{10}$

$f'''(x) = e^x$ $\quad f'''(10) = e^{10}$

$$\therefore f(x) = f(10) + \frac{(x-10) f'(10)}{1!} + \frac{(x+10)^2 f''(10)}{2!} + \cdots + \frac{f^n(10)(x-10)^n}{n!}$$

$$= e^{10} + (x+10)e^{10} + \frac{(x+10)^2 e^{10}}{2!} + \frac{(x+10)^3 e^{10}}{3!} + \cdots$$

Example; find the Taylor's Series expansion of $f(x) = e^{3x}$ at $x=0$. Hence find $e^{0.3}$

Soln

$f(x) = e^{3x}$

$f'(x) = 3e^{3x}$ $\qquad f'(0) = 3$

$f''(x) = 9e^{3x}$ $\qquad f''(0) = 9$

$f'''(x) = 27e^{3x}$ $\qquad f'''(0) = 27$

$f^{iv}(x) = 81e^{3x}$ $\qquad f^{iv}(0) = 81$

$$e^{3x} = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{iv}(0)\frac{x^4}{4!} + \cdots$$

$$= 1 + 3x + \frac{9x^2}{2} + \frac{27x^3}{3!} + \frac{81x^4}{4!} + \cdots$$

$$= 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8} + \cdots$$

To calculate $e^{0.3}$, we equate

$$e^{3x} = e^{0.3} \implies 3x = 0.3 \implies x = 0.1$$

by substitution into the series, we obtain

$$e^{0.3} = 1 + 3(0.1) + \frac{9(0.1)^2}{2} + \frac{9(0.1)^3}{2} + \frac{27(0.1)^4}{8} + \cdots$$

$$= 1.3498.$$

__Example;__ $f(x) = \sin 2x$ at $x = 0$

__Soln.__

$f(x) = \sin 2x$ $\qquad f(0) = 0$

$f'(x) = 2\cos 2x$ $\qquad f'(0) = 2$

$f''(x) = -4\sin 2x$ $\qquad f''(0) = 0$

$f'''(x) = -8\cos 2x$ $\qquad f'''(0) = -8$

$f^{iv}(x) = 16\sin 2x$ $\qquad f^{iv}(0) = 0$

$$f(x) = \sin 2x = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{iv}(0)\frac{x^4}{4!}$$

$$= 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!}$$

$$= 2x - \frac{4x^3}{3} + \frac{4}{15}x^5$$

Example; find the Taylors series expansion for $f(x) = \ln(1+x)$; use it to evaluate $\ln(1.02)$ to 4 d.p

Soln

$f(x) = \ln(1+x)$ $\quad f(0) = \ln 1 = 0$

$f'(x) = \frac{1}{1+x}$ $\quad f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2}$ $\quad f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3}$ $\quad f'''(0) = 2$

$f^{iv}(x) = \frac{-6}{(1+x)^4}$ $\quad f^{iv}(0) = -6$

$$\ln(1+x) = 1 \cdot x + (-1)\frac{x^2}{2} + \frac{2x^3}{3!} + \frac{(-6)x^4}{4!}$$

$$= x - \frac{x^2}{2} + \frac{2x^3}{3!} - \frac{6x^4}{4!}$$

$$\ln(1+x) = \ln(1.02) \implies x = 0.02$$

$$\ln(1.02) = \ln(1+0.02) = 0.02 - \frac{(0.02)^2}{2} + \frac{2(0.02)^3}{3!} - \frac{6(0.02)}{4!}$$

$$= 0.0198.$$

Exercises; (A) Show that (1) for a small value of $x$

$$(1+2x)e^{-x} + \log_e^{(1+2x)} \simeq 1 + 3x - \frac{7x^2}{2} + \frac{7x^3}{2}$$

(2) $\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \frac{1}{9}x^9 + \ldots\right)$

(3) for $x > 1$, $\log_e^{\left(\frac{x+1}{x-1}\right)} = 2\left(\frac{1}{x} + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \ldots\right)$,

evaluate $\log_e^2$ (i.e $x = 2$)

(4) $\log_e^{(3+4x)} = \log_e^3 + \frac{4}{3}x - \frac{8}{9}x^2 + \frac{64}{81}x^3 - \ldots$ and state the limits between which $x$ must be for the expansion to be valid.

(5) $\log_e \sec^2 x = \tan^2 x - \frac{\tan^4 x}{2} + \frac{\tan^6 x}{3} - \frac{\tan^8 x}{4} + \dots$

(6) Write down the first few terms of the maclaurin's Series for (a) $e^{2x}$ (b) $e^{-3x}$ (c) $e^{x^2}$ (d) $\log_e(1+2x)$ (e) $\log_e(1-3x)$ (f) $\log_e(1+x^2)$ (g) $\log_e(1-x^2)$

(7) If $x$ is so small that $x^4$ and higher powers of $x$ can be neglected, show that $e^x + \log_e(1-x) = 1 - \frac{1}{6}x^3$ approximately.

Integration, which can be referred to as the reverse operation of differentiation involves determining the original function from its derivative.

$\int f(x)\,dx$ denotes the antiderivative of $f(x)$ called the Integral of $f(x)$ with respect to $x$. where $f(x)$ is the Integrand.

Standard forms

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$
2. $\int 0\,dx = c$
3. $\int 1\,dx = x + c$
4. $\int K\,dx = Kx + c$
5. $\int \sin x\,dx = -\cos x + c$
6. $\int \cos x = \sin x + c$
7. $\int \sec^2 x\,dx = \tan x + c$
8. $\int \tan x \sec x\,dx = \sec x + c$
9. $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$
10. $\int \cot \operatorname{cosec}\,dx = -\operatorname{cosec} x + c$
11. $\int e^x\,dx = e^x + c$
12. $\int \frac{f'(x)}{f(x)}\,dx = \log_e f(x) + c$
13. $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Note that; c is the constant of integration.

Examples; Evaluate the ff.

1. $\int x^4\,dx$ 2. $\int 4x^5\,dx$ 3. $\int 3x\,dx$ 4. $\int x^{3/4}\,dx$
5. $\int \frac{1}{x^3}\,dx$ 6. $\int 4\,dx$ 7. $\int \frac{1}{3\sqrt{x}}\,dx$ 8. $\int (2x^4 - x + 5)\,dx$
9. $\int \frac{x^4 + x - 3}{x^3}\,dx$ 10. $\int (x-2)^2\,dx$ 11. $\int \left(\frac{1}{x^2} + \frac{2}{3\sqrt{x}}\right)dx$
12. $\int (e^x + \sec^2 x)\,dx$ 13. $\int \left(\cos x - \frac{1}{x}\right)dx$.

Soln

1. $\int x^4\,dx = \frac{x^5}{5} + c$ where c is the constant of integration

3. $\int 3x\,dx = \frac{3x^2}{2} + c$

(6) $\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-2}}{-2} + c = \frac{-1}{2x^2} + c$

(7) $\int \frac{1}{3\sqrt{x}}\,dx = \frac{1}{3}\int x^{-\frac{1}{2}}\,dx = \frac{1}{3}\cdot 2x^{1/2} + c = \frac{2}{3}x^{1/2} + c$

(9) $\int \frac{x^4 + x - 3}{x^3}\,dx = \int\left(\frac{x^4}{x^3} + \frac{x}{x^3} - \frac{3}{x^3}\right)dx$

$= \int\left(x + \frac{1}{x^2} - \frac{3}{x^3}\right)dx = \int (x + x^{-2} - 3x^{-3})$

$= \frac{x^2}{2} - \frac{1}{x} + \frac{3}{2x^2} + c$.

(13) $\int \left(\cos x - \frac{1}{x}\right)dx = \sin x - \ln x + c$

*Solve the remaing as an Exercises!

## The Substitution Rule of Integration

Evaluate the following by making an appropriate Substitution.

(1) $\int (\sqrt{2x} - 1)\,dx$ (2) $\int e^{-x^3} x^2\,dx$ (3) $\int \frac{x^3\,dx}{(3x^4 - 5)^6}$

(4) $\int 18x^2 \sqrt[4]{6x^3 + 5}\,dx$ (5) $\int \left(1 - \frac{1}{y}\right)\cos(y - \ln y)\,dy$ (6) $\int \frac{2\,dx}{x\ln x}$

(7) $\int \sec^2\left(\frac{2x - 3}{5}\right)dx$ (8) $\int \frac{x^2}{(x^3 + 7)^{2/3}}\,dx$ (9) $\int x^2(3 - 10x^3)^4$

(10) $\int (y - 4)^2 \cos(y - 4)^3\,dy$ (11) $\int \frac{\sin^{-1}x}{\sqrt{1 - x^2}}\,dx$ (12) $3\tan^2 4x\,dx$

(13) $\int \sin^3 x\,dx$ (14) $\int x^2 \sqrt[4]{1 - 2x^3}\,dx$ (15) $x^2\sqrt{1 + x^2}\,dx$

(16) $\int e^{2 - 5x}\,dx$ (17) $\int \frac{x^3}{\sqrt[3]{x^4 - 5}}\,dx$ (18) $\int \tan x\,dx$ (19) $\int \sec x\,dx$

(20) $\int \cot x\,dx$ (21) $\int \operatorname{cosec} x\,dx$ (22) $\int \frac{\sin x}{1 - \cos x}\,dx$

(23) $\int \frac{x}{2x^2 + 1}\,dx$ (24) $\int \frac{dx}{(x - 3)^2}$ (25) $\int \frac{2^x - 2^{-x}}{2^x + 2^{-x}}\,dx$

(26) $\int \frac{(x + 3)\,dx}{\sqrt{x^2 + 6x - 3}}$ (27) $\int \operatorname{cosec}^2 5x\,dx$ ~~(28) $\int e^{2x}\,dx$~~

(28) $\int \frac{e^{2x}\,dx}{\sqrt{e^x + 1}}$.

① $\int \sqrt{2x-1}\, dx$

Let, $u = 2x-1$

$\frac{du}{dx} = 2 \implies dx = \frac{du}{2}$;

Substituting back into the integral, we have

$$\int \sqrt{2x-1}\, dx = \int \sqrt{u}\, \frac{du}{2} = \frac{1}{2}\int u^{\frac{1}{2}} du = \frac{1}{2}\frac{u^{3/2}}{3/2}$$

$$= \frac{1}{3}u^{3/2} = \frac{1}{3}(2x-1)^{3/2} + c$$

② $\int e^{-x^3} x^2\, dx$

Let $u = -x^3$

$\frac{du}{dx} = -3x^2 \implies dx = -\frac{du}{3x^2}$

Substituting gives,

$$\int e^{u} x^2 \cdot \frac{du}{-3x^2} = -\frac{1}{3}\int e^{u}\, du = -\frac{1}{3}e^{u} + c$$

$$= -\frac{1}{3}e^{-x^3} + c$$

⑰ $\int \frac{x^3}{\sqrt[3]{x^4+5}}\, dx$

Let $u = x^4+5$

$\frac{du}{dx} = 4x^3 \implies dx = \frac{du}{4x^3}$

Substituting gives,

$$\int \frac{x^3}{\sqrt[3]{x^4+5}}\, dx = \int \frac{x^3}{\sqrt[3]{u}} \cdot \frac{du}{4x^3} = \frac{1}{4}\int u^{-1/3}\, du$$

$$= \frac{1}{4} \cdot u^{2/3} \cdot \frac{3}{2} + c$$

$$= \frac{3}{8}(x^4+5)^{2/3} + c$$

⑱ $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$

Let $u = \cos x$ so that

$\frac{du}{dx} = -\sin x \implies dx = \frac{-du}{\sin x}$

Substituting gives

$$\int \frac{\sin x}{\cos x}\, dx = \int \frac{\sin x}{u} \cdot \frac{-du}{\sin x} = -\int \frac{du}{u} = -\ln u + c$$

$$= -\ln(\cos x) + c = \ln(\cos x)^{-1} + c = \ln \sec x + c$$

... of linear algebraic functions.
(Derivative of 1st degree).

① $\int \frac{2x^3 - x^2 - x}{2x-3}\, dx = \int \left(x^2 + x + 1 + \frac{3}{2x-3}\right) dx$

use long division

$= \frac{x^3}{3} + \frac{x^2}{2} + x + \frac{3}{2}\log_e(2x-3) + C$

② $\int \frac{7 + x - 2x^2}{2-x}\, dx$

$= \int \left(2x + 3 + \frac{1}{2-x}\right) dx = x^2 + 3x - \log_e(2-x) + C$

③ $\int \left(\frac{x+3}{x-2}\right) dx = \int \left(1 + \frac{5}{x-2}\right) dx$

$= x + 5\ln(x-2) + C.$

## Definite Integrals.

① $\int_0^1 x^2\, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - \frac{0}{3} = \frac{1}{3}.$

② $\int_{-3}^{-2} \frac{dx}{x^3} = \int_{-3}^{-2} x^{-3}\, dx = \left[-\frac{1}{2}\cdot\frac{1}{x^2}\right]_{-3}^{-2}$

$= -\frac{1}{2}\left[\frac{1}{4} - \frac{1}{9}\right] = -\frac{1}{2} \times \frac{5}{36}$

$= \frac{-5}{72}.$

Evaluate the following

① $\int \frac{x^2}{x-1}\, dx$ ② $\int \frac{x}{x-1}\, dx$ ③ $\int \frac{t}{1-3t}\, dt$ ④ $\int \frac{t^2}{1-3t}\, dt$

⑤ $\int_0^{\frac{1}{2}} \frac{2\theta - 3\theta^2}{1-\theta}\, d\theta$ ⑥ $\int_0^1 \frac{2x - 8x^2}{1+4x}\, dx$ ⑦ $\int \left(\frac{2x+3}{5x-4}\right) dx$

⑧ $\int \frac{3x^2}{4x+1}\, dx$ ⑨ $\int \frac{8x^2}{3x-5}\, dx$ ⑩ $\int \frac{4x-7}{5x+3}\, dx.$

# TRIGONOMETRIC SUBSTITUTIONS.

If the Integral Involves.

i) $\sqrt{a^2-x^2}$, use $x = a\sin\theta$

ii) $\sqrt{a^2+x^2}$, use $x = a\tan\theta$

iii) $\sqrt{x^2-a^2}$, use $x = a\sec\theta$.

**Examples;**

$$\int_2^{2\sqrt{3}} \frac{dx}{x^2\sqrt{4+x^2}}$$

let $x = 2\tan\theta$, then $\frac{dx}{d\theta} = 2\sec^2\theta$

when $x=2$, $\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$x = 2\sqrt{3}$, $\tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$

Now; $\displaystyle\int_2^{2\sqrt{3}} \frac{dx}{x^2\sqrt{4+x^2}} = \int_{\pi/4}^{\pi/3} \frac{2\sec^2\theta}{4\tan^2\theta\, 2\sec\theta}\, d\theta$

$$= \frac{1}{4}\int_{\pi/4}^{\pi/3} \frac{\sec\theta}{\tan^2\theta}\, d\theta = \frac{1}{4}\int_{\pi/4}^{\pi/3} \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{4}\int_{\pi/4}^{\pi/3} \operatorname{cosec}\theta\cot\theta\, d\theta = \frac{1}{4}\Big[-\operatorname{cosec}\theta\Big]_{\pi/4}^{\pi/3}$$

$$= \frac{1}{4}\left(\frac{-2}{\sqrt{3}} + \sqrt{2}\right) = 0.005$$

**Example;** $\displaystyle\int \frac{\sqrt{9-4x^2}}{x}\, dx$

**Soln**

Let $2x = 3\sin\theta$ So that $dx = \frac{3}{2}\cos\theta\, d\theta$

$(2x)^2 = 3^2\sin^2\theta$ and $9-4x^2 = 9(1-\sin^2\theta)$

$= 9\cos^2\theta$.

$\sqrt{9-4x^2} = \sqrt{9\cos^2\theta} = 3\cos\theta$

$$\int \frac{\sqrt{9-4x^2}}{x}\, dx = \int \frac{3\cos\theta \cdot \frac{3}{2}\cos\theta\, d\theta}{\frac{3}{2}\sin\theta}$$

$$\Rightarrow 3\int \frac{\cos^2\theta}{\sin\theta}\, d\theta = 3\int \frac{1-\sin^2\theta}{\sin\theta}\, d\theta$$

$\therefore 3\int(\csc\theta - \sin\theta)\,d\theta = 3\ln|\csc\theta - \cot\theta| + 3\cos\theta +$

__Example;__ $\displaystyle\int_0^2 \sqrt{\left(\frac{x}{4-x}\right)}\,dx$

__Soln__

Let $x = 4\sin^2\theta$. $\left[\text{ans } \pi - 2\right]$. try it

__Example;__ $\displaystyle\int_0^a \sqrt{a^2 - x^2}\,dx$

__Soln__

Let $x = a\sin\theta$. $\left[\text{ans } \dfrac{\pi a^2}{4}\right]$ try it also

__Exercises;__

① $\displaystyle\int \frac{dx}{\sqrt{25-16x^2}}$ ② $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+5}}$ use $x+1 = 2\tan\theta$.

③ $\displaystyle\int \frac{dx}{x^2+a^2}$ ④ $\displaystyle\int \frac{x^2}{\sqrt{x^2-9}}\,dx$ ⑤ $\displaystyle\int x\sqrt{x^2-9}$

⑥ $\displaystyle\int x\sqrt{x+1}\,dx$ ⑦ $\displaystyle\int x\sqrt{x^2+4}$ ⑧ $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\,dx$ let $x = \cos 2\theta$

⑨ $\displaystyle\int x^2\sqrt{x-1}$ ⑩ $\displaystyle\int \frac{dx}{x^2+10x+30}$ ⑪ $\displaystyle\int \frac{(x+3)}{\sqrt{1-x^2}}\,dx$

⑫ $\displaystyle\int \frac{dx}{\sqrt{x^2-4x+13}}$ ⑬ $\displaystyle\int \frac{x+3}{5-4x-x^2}\,dx$

## Trigonometric Integrands.

Example; Evaluate ① $\int \sin^2 x\,dx$ ② $\int \cos^2 x\,dx$

__Soln__

① $\displaystyle\int \sin^2 x\,dx = \int \tfrac{1}{2}(1-\cos 2x)\,dx = \tfrac{1}{2}\left(x - \tfrac{1}{2}\sin 2x\right) + c$

② $\displaystyle\int \cos^2 x\,dx = \int \tfrac{1}{2}(1+\cos 2x)\,dx = \tfrac{1}{2}\left(x + \tfrac{1}{2}\sin 2x\right) + c$

Evaluate; $\int \sin^2 x \cos x \, dx$

Soln

Let $u = \sin x$ So that $du = \cos x \, dx$

$$\int \sin^2 x \cos x \, dx = \int u^2 \, du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C$$

Evaluate; $\int \cos^3 x \, dx$

Soln.

$$\int \cos^3 x \, dx = \int (1 - \sin^2 x) \cos x \, dx$$
$$= \int \cos x \, dx - \int \sin^2 x \cos x \, dx$$
$$= \sin x - \frac{\sin^3 x}{3} + C$$

Evaluate; $\int \tan^3 x \sec x \, dx$

Soln.

This gives

$$\int \tan^2 x \tan x \sec x \, dx = \int (1 - \sec^2 x) \tan x \sec x \, dx$$

Let $u = \sec x$ So that $du = \sec x \tan x \, dx$; Then

$$\int (1 - \sec^2 x) \tan x \sec x \, dx = \int (1 - u^2) \, du$$
$$= u - \frac{u^3}{3} + C = \sec x - \frac{\sec^3 x}{3} + C$$

Note; if the integrand is a product of a sine and/or a cosine of a multiple angle, the identities below are applied.

1. $\sin px \cos qx = \frac{1}{2} [\sin(p+q)x + \sin(p-q)x]$
2. $\sin px \sin qx = \frac{1}{2} [\cos(p-q)x - \cos(p+q)x]$
3. $\cos px \cos qx = \frac{1}{2} [\cos(p-q)x + \cos(p+q)x]$.

Example;

1. $$\int \sin 6x \cos 2x \, dx = \int \frac{1}{2} \left( \sin(6+2)x + \sin(6-2)x \right) dx$$
$$= \int \frac{1}{2} \left( \sin 8x + \sin 4x \right) dx$$
$$= -\frac{1}{16} \left( \cos 8x + 2 \cos 4x \right) + C$$

② $\int \sin 6x \sin 2x = \int \frac{1}{2}(\cos 4x \quad - \quad )$

$= \int \frac{1}{2}\left(\frac{1}{4}\sin 4x - \frac{1}{8}\sin 8x\right) + C$

$= \frac{1}{16}(2\sin 4x - \sin 8x) + C$.

## Integration by Partial fraction.

Difference cases are considered here depending on the nature of the factor of the denominator (of the integrand)

Evaluate $\int \frac{3x-1}{(x+1)(x-2)}\,dx$

Soln.

$\int \frac{3x-1}{(x+1)(x-2)}\,dx \Rightarrow \frac{3x-1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} = \frac{A(x-2)+B(x+1)}{(x+1)(x-2)}$

So that $3x-1 = A(x-2) + B(x+1)$.

Using the substitution $x=-1$ and $x=2$ gives

$A = 4/3$ and $B = 5/3$.

$\int \frac{3x-1}{(x+1)(x-2)}\,dx = \int \frac{4}{3(x+1)}\,dx + \int \frac{5}{3(x-2)}\,dx$

$= \frac{4}{3}\ln(x+1) + \frac{5}{3}\ln(x-2) + C$

Evaluate $\int \frac{2x-1}{(1-x)(1+x)^2}\,dx$

Soln.

$\frac{2x-1}{(1-x)(1+x)^2} = \frac{A}{(1-x)} + \frac{B}{(1+x)} + \frac{C}{(1+x)^2} = \frac{A(1+x)^2 + B(1-x)(1+x) + C(1-x)}{(1-x)(1+x)^2}$

$\Rightarrow$

$2x-1 = A(1+x)^2 + B(1-x)(1+x) + C(1-x)$

$2x-1 = A(1+2x+x^2) + B(1-x^2) + C(1-x)$

for $x=1 \Rightarrow 1 = 4A \Rightarrow A = 1/4$

for $x=-1 \Rightarrow -3 = 2C \Rightarrow C - \frac{1}{2}$.

Comparing coeff of $x^2$

$0 = A - B \Rightarrow A = B$ or $B = \frac{1}{4}$

$$\int \frac{2x-1}{(1-x)(1+x)^2}\,dx = \int \frac{1}{4(1-x)}\,dx + \int \frac{1}{4(1+x)}\,dx + \int \frac{3}{2(1+x)^2}\,dx$$

$$= -\frac{1}{4}\ln(1-x) + \frac{1}{4}(1+x) + \frac{3}{2}(1-x)^{-1} + C$$

$$= \frac{1}{4}\ln\frac{(1+x)}{(1-x)} + \frac{3}{2(1-x)} + C.$$

Example; Evaluate $\int \frac{(3x+1)}{x^3+2x^2+x+2}\,dx$

Soln.

$$\int \frac{3x+1}{x^3+2x^2+x+2}\,dx = \int \frac{3x+1}{(x^2+1)(x+2)}\,dx$$

Set

$$\frac{3x+1}{(x^2+1)(x+2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+2}.$$

Therefore

$3x+1 = (Ax+B)(x+2) + C(x^2+1)$

with $x = -2$

$-5 = 5C \Rightarrow C = -1$.

equating coefficient of $x^2$ gives $0 = A + C \Rightarrow A = 1$

equating constants gives $1 = 2B + C \Rightarrow B = 1$

$$\int \frac{3x+1}{(x^2+1)(x+2)}\,dx = \int \frac{x+1}{x^2+1}\,dx - \int \frac{1}{x+2}\,dx$$

$$= \int \frac{x}{x^2+1}\,dx - \int \frac{dx}{x^2+1}\,dx - \int \frac{dx}{x+2}\,dx$$

$$= \frac{1}{2}\log_e(x^2+1) - \arctan x - \log_e(x+2) + C$$

Exercises, Evaluate the following integrals.

① $\int \frac{2x^2-10x}{(x+3)(x-1)^2}\,dx$ ② $\int \frac{e^t\,dt}{e^{2t}+3e^t+2}$ ③ $\int \frac{4x^2-2x-7}{2x^2-3x-2}\,dx$

④ $\int \frac{4x^2-3x+5}{(x+2)(x-1)^2}\,dx$ ⑤ $\int \frac{x+62}{(3x-1)^2(2x+3)}\,dx$ ⑥ $\int \frac{4x}{(x^2+4)(x^2+8)}\,dx$

⑦ $\int \frac{8x^2+3x-3}{(2x^2-1)(2x+3)}\,dx$ ⑧ $\int \frac{10}{(x-1)(x^2+9)}\,dx$ ⑨ $\int \frac{x^4}{x^2-9}\,dx$

⑩ $\int \frac{6x^3+10x^2-13x-6}{3x^3+x^2}\,dx$ ⑪ $\int \frac{dx}{x^4-1}$ ⑫ $\int \frac{x}{x^2-3x-4}\,dx$

## Integration by parts.

From product rule;

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$, on Integration we obtain

$$\int u\,dv = uv - \int v\,du$$

Evaluate $\int x^2e^{-x}\,dx$

Soln

Let $u = x^2$, $dv = e^{-x}\,dx$

$du = 2x\,dx$ $v = -e^{-x}$

using $\int u\,dv = uv - \int v\,du$ we have;

$$I = \int x^2e^{-x}\,dx = -x^2e^{-x} - \int -e^{-x}\,2x\,dx$$
$$= -x^2e^{-x} + 2\int xe^{-x}\,dx$$

In the 2nd integral, let $u = x$, $dv = e^{-x}\,dx$
$du = dx$, $v = -e^{-x}$

Substituting gives

$$I = -x^2e^{-x} + 2\left[-xe^{-x} - \int -e^{-x}\,dx\right]$$
$$= -x^2e^{-x} + 2xe^{-x} - 2e^{-x} + C$$
$$= -e^{-x}\left[x^2 + 2x + 2\right] + C$$

Evaluate $\int e^x \cos x\,dx$

Soln

Let $u = e^x$; $dv = \cos x\,dx$

$du = e^x$; $v = \sin x$

using $\int u\,dv = uv - \int v\,du$ we get

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx \quad \text{--- (1)}$$

appling integration by parts on the integral on the right.

i.e $\int e^x \sin x\,dx \Rightarrow$ $u = e^x$; $dv = \sin x\,dx$

$du = e^x\,dx$ $v = -\cos x$

$$\int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx \quad \text{--- (2)}$$

Substituting (2) in (1) gives.

$$\int e^x \cos x\,dx = e^x \sin x - [-e^x \cos x + \int e^x \cos x\,dx]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx + c$$

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x + C$$

$$\int e^x \cos x\,dx = \frac{1}{2}(e^x \sin x + e^x \cos x) + c$$

$$= \frac{e^x}{2}(\sin x + \cos x) + c.$$

Evaluate; $\int \tan^{-1} x\,dx$

Soln.

Using $\int u\,dv = uv - \int v\,du$

$u = \tan^{-1} x$; $dv = dx$

$du = \frac{1}{1+x^2}dx$; $v = x$

$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \frac{x}{1+x^2}dx$$

$$= x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + c.$$

Exercises, Evaluate ...

① $\int \ln x \, dx$ ② $\int \sin^{-1} x \, dx$ ③ $\int \sec^3 x \, dx$ ④ $\int x^2 \sin x \, dx$

⑤ $\int x \tan^2 x \, dx$ ⑥ $\int x^2 \ln x \, dx$ ⑦ $\int e^x \sin x \, dx$ ⑧ $\int x \sin^{-1} x$

⑨ $\int_0^{\pi/2} x \cos 2x \, dx$ ⑩ $\int x \tan^{-1} x \, dx$ ⑪ $\int_0^{\pi/4} \sec^2 x \, dx$.

# Reduction Formula.

Example, obtain a reduction formula for

i) $\int x^n e^x \, dx$; $\forall n \geq 2$

Soln.

Let $I_n = \int x^n e^x \, dx$

from Integration by parts.

$u = x^n$ ; $dv = e^x \, dx$

$du = n x^{n-1} \, dx$, $v = -e^x$

$$I_n = x^n e^x - n \int x^{n-1} e^x \, dx$$

$$= x^n e^x - n I_{n-1}$$

note that, we stop at $I_0 = e^x$.

② $\int x^3 e^x \, dx = x^3 e^x - 3 \int x^2 e^x \, dx$

$$= x^3 e^x - 3 \left[ x^2 e^x - 2 \int x e^x \, dx \right]$$

$$= x^3 e^x - 3x^2 e^x + 6 \left[ x e^x - \int e^x \, dx \right].$$

③ $\int \sin^n x \, dx$

Soln.

Let $I_n = \int \sin^n x \, dx = \int \sin^{n-1} x \sin x \, dx$

let $u = \sin^{n-1} x$ $dv = \sin x \, dx$

$dv = (n-1) \sin^{n-2} x \cos x \, dx$ and $v = -\cos x$

from $\int u \, dv = uv - \int v \, du$

$$I_n = -\sin^{n-1}x\cos x + (n-1)\int \cos^2 x \sin^{n-2}x\,dx$$

$$= -\sin^{n+1}x\cos x + (n-1)\int (1-\sin^2 x)\sin^{n-2}x\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$

$$I_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

Combining the terms with $I_n$, we get

$$I_n = \frac{1}{n}\left[-\sin^{n-1}x\cos x + (n-1)I_{n-2}\right], \quad n \geq 2.$$

(1) Show that $\int \sin^4 x\,dx = -\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\sin x\cos x + \frac{3}{8}x + C$.

using the above.

(2) Show that if $I_n = \int \cos^n x\,dx$, Then

$$I_n = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}I_{n-2}.$$

(3) If $I_n = \int \sec^n x\,dx$ Then

$$I_n = \int \sec^2 x\cdot\sec^{n-2}x\,dx =$$

$$= \sec^{n-2}x\tan x - (n-2)\int \sec^{n-2}x\tan^2 x\,dx$$

$$= \frac{1}{n-1}\left(\sec^{n-2}x\tan x + (n-2)I_{n-2}\right).$$

Reduction formula for $\int \sin^m x\cos^n x\,dx$ where $m,n \geq 2$.

$$I_{(m,n)} = \int \sin^m x\cos^n x\,dx$$

$$= \int \sin^m x\cos^{n-1}x\cos x\,dx \quad \text{or} \quad \int \sin^{m-1}x\cos^n x\sin x\,dx$$

Let $u = \cos^{n-1}x\,dx$; $dv = \sin^m x\cos x\,dx$

$\frac{du}{dx} = (n-1)\cos^{n-2}x(-\sin x)$ $\quad v = \int \sin^m x\cos x\,dx$

To find $v$

$dx = du$

$$\Rightarrow V = \frac{1}{m+1}\sin^{m+1}x.$$

So that,

$$I_{(m,n)} = \cos^{n-1}x\left(\frac{1}{m+1}\sin^{m+1}x\right) - \int \frac{1}{m+1}\sin^{m+1}x\,(n-1)\cos^{n-2}x\,(-\sin x)\,dx$$

$$= \frac{1}{m+1}\cos^{n-1}x\sin^{m+1}x + \frac{n-1}{m+1}\int \sin^{m+2}x\cos^{n-2}x\,dx$$

$$= \frac{1}{m+1}\cos^{n-1}x\sin^{m+1}x + \frac{n-1}{m+1}\int \sin^{m}x(\sin^2 x)\cos^{n-2}x\,dx$$

$$= \frac{1}{m+1}\cos^{n-1}x\sin^{m+1}x + \frac{n-1}{m+1}\int \sin^{m}x(1-\cos^2 x)\cos^{n-2}x\,dx$$

$$= \frac{1}{m+1}\cos^{n-1}x\sin^{m+1}x + \frac{n-1}{m+1}\int \sin^{m}x\cos^{n-2}x\,dx - \frac{n-1}{m+1}\int \sin^m x\cos^n x\,dx$$

Therefore,

$$I_{(m,n)} = \frac{1}{m+1}\cos^{n-1}x\sin^{m+1}x + \frac{n-1}{m+1}I_{(m,n-2)} - \frac{n-1}{m+1}I_{(m,n)}.$$

$$I_{(m,n)}\left(\frac{m+1+(n-1)}{m+1}\right) = \frac{1}{m+1}\cos^{n-1}x\sin^{m+1}x + \frac{n-1}{m+1}I_{(m,n-2)}.$$

$$I_{(m,n)}\left(\frac{m+n}{m+1}\right) = \frac{1}{m+1}\left(\cos^{n-1}x\sin^{m+1}x + (n-1)I_{(m,n-2)}\right).$$

$$I_{(m,n)} = \frac{1}{m+n}\left(\cos^{n-1}x\sin^{m+1}x + (n-1)I_{(m,n-2)}\right).$$

Alternatively; if we set

$$I_{(m,n)} = \int \sin^m x\cos^n x\,dx = \int \sin^{m-1}x\cos^n x\sin x\,dx$$

choose $u = \sin^{m-1}x$ and

$dv = \cos^n x\sin x\,dx$

Then,

$$I_{(m,n)} = \frac{-1}{m+1}\left(\sin^{m-1}x\cos^{n+1}x - (m-1)I_{(m-2,n)}\right)$$ check!!!

<u>Example</u>; using the above reduction formula find

$$\int \sin^2 x\cos^4 x\,dx$$

Sol<sup></sup>

$\int \sin^2 x \cos^4 x \, dx$.

$\Rightarrow m=2, \quad n=4$ Using $I_{(m,n)} = \frac{1}{m+n}\left(\cos^{n-1}x \sin^{m+1}x + (n-1) I_{(m,n-2)}\right)$

we have;

$I_{(2,4)} = \frac{1}{6}\left(\cos^3 x \sin^3 x + 3 I_{(2,2)}\right)$

$= \frac{1}{6}\left(\cos^3 x \sin^3 x + 3\left(\frac{1}{4}\cos x \sin^3 x + I_{(2,0)}\right)\right)$

where $I_{(2,0)} = \int \sin^2 x \, dx = \frac{1}{2}\int (1-\cos 2x)\,dx$

$= \frac{x}{2} - \frac{\sin 2x}{4}$.

Thus;

$I_{(2,4)} = \frac{1}{6}\left((\cos x \sin x)^3 + 3\left(\frac{1}{4}\cos x \sin^3 x + \frac{x}{2} - \frac{\sin 2x}{4}\right)\right)$

$= \frac{1}{6}\left(\cos^3 x \sin^3 x + \frac{3}{4}\cos x \sin^3 x + \frac{3x}{2} - \frac{3}{4}\sin 2x\right)$

Exercise; obtain the reduction formula for the f/f.

(a) $\int \tan^n x \, dx$ (b) $\int \cot^n x \, dx$ (c) $\int \sec^n x \, dx$ (d) $\int \operatorname{cosec}^n x \, dx$

(2) Show that i) if $I_{(m,n)} = \int \sec^n x \tan^m x \, dx$, then

$I_{(n,m)} = \frac{1}{n+m-1}\sec^n x \tan^{m-1} x - \frac{m-1}{n+m-1} I_{(n,m-2)}$.

or

$I_{(n,m)} = \frac{1}{n+m-1}\left(\sec^{n-2} x \tan^{m+1} x + (n-2) I_{(n-2,m)}\right)$.

Hint; $\int \sec^n x \tan^m x \, dx = \int \sec^{n-1} x \tan^{m-1} x \,(\sec x \tan x)\, dx$

let $u = \sec^{n-1} x \tan^{m-1} x$

$dv = \sec x \tan x \, dx$.

Similar approaches in (1) above.

Q. $I_{(n,m)} = \int \operatorname{cosec}^n x \cot^m x \, dx$, show that

$I_{(n,m)} = \frac{-1}{n+m-1}\left(\operatorname{cosec}^n x \cot^{m-1} x + (m-1) I_{(n,m-2)}\right)$ for $n,m \geq 1$

$= \frac{-1}{n+m-1}\left(\operatorname{cosec}^{n-2} x \cot^{m-1} x + (n-2) I_{(n-2,m)}\right)$.

# t - Formula.

Evaluate (1) $\int \frac{1}{2\sin^2 x + 4\cos^2 x}\,dx$,

Soln,

$t = \tan x$, $\sin x = \frac{t}{\sqrt{1+t^2}}$; $\cos x = \frac{dt}{1+t^2}$,

$2\sin^2 x + 4\cos^2 x = \frac{2t^2}{1+t^2} + \frac{4}{1+t^2} = \frac{2t^2+4}{1+t^2}$.

$\therefore \int \frac{1}{2\sin^2 x + 4\cos^2 x}\,dx = \int \frac{1+t^2}{2t^2+4}\cdot\frac{dt}{1+t^2} = \frac{1}{2}\int\frac{1}{t^2+2}\,dt$

Let $t = \sqrt{2}\tan\theta$ and ~~$dt = \sqrt{2}\sec$~~ $dt = \sqrt{2}\sec^2\theta\, d\theta$

Then,

$$\int \frac{1}{2\sin^2 x + 4\cos^2 x}\,dx = \frac{1}{2}\int\frac{1}{t^2+2}\,dt$$

$$= \frac{1}{2}\,\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + C.$$

(2) $\int \frac{1}{1+\sin x - \cos x}\,dx$

Soln,

$t = \tan\frac{x}{2}$, $\sin\frac{x}{2} = \frac{t}{\sqrt{1+t^2}}$; $\cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$.

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{t}{\sqrt{1+t^2}}\cdot\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$.

$\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} \Rightarrow dx = \frac{2dt}{1+t^2}$,

$1 + \sin x - \cos x = 1 + \ldots 2t \ldots - \ldots t^2$

$$1+\sin x-\cos x = \frac{1+t^2+2t-1+t^2}{1+t^2} = \frac{2(t^2+t)}{1+t^2}$$

$$\int \frac{1}{1+\sin x-\cos x}\,dx = \int \frac{1+t^2}{2(t^2+t)}\cdot\frac{2\,dt}{1+t^2} = \int \frac{1}{t^2+t}\,dt$$

$$= \int\left(\frac{1}{t}-\frac{1}{1+t}\right)dt$$

$$= \ln t - \ln(1+t) + C$$

$$= \ln\left(\frac{t}{1+t}\right)+c$$

$$= \ln\left(\frac{\tan x/2}{1+\tan x/2}\right)+c.$$

Exercises; Evaluate the f/f using t- formula.

① $\int \frac{dx}{1+3\sin x+\cos x}$ ② $\int \frac{dx}{1+\sin x}$ ③ $\int \frac{dx}{\sin x+\tan x}$

④ $\int \frac{dx}{\sin x-\cos x}$ ⑤ $\int \frac{\cos x}{2-\cos x}\,dx$.

## Real-valued functions of two or Three Variables.

A Variable $Z$ is Said to be a function of two Variable $x$ and $y$ if for each given pair $(x,y)$ we can determine one or more Values of $Z$. we use the notation $f(x,y)$, $F(x,y)$ etc to denote the values of the functions at $(x,y)$ and write $Z=f(x,y)$, $Z=F(x,y)$ for the variables $x, y, w$, we write $Z=f(x,y,w)$.

Examples;

If ~~f(x)~~ $f(x,y)=x^2+2y^3$ then

$f(3,1)=3^2+2(1)^3 = 9+2 = 11$.

(2) If $f(x,y) = yx - 2x^2y^3$ Then.
$f(1,2) = (1)(2) - 2(1)^2(2)^3 = 2 - 16 = -14$.

## PARTIAL DERIVATIVES.

For a single variable function, $y = f(x)$, its derivative is only w.r.t the independent variable $x$. But if the function of two or more variables say $f(x,y,z)$, we can either find its derivative "partially" w.r.t $x$ (treating $y, z$ as constants) w.r.t $y$ (treating $x$ and $z$ as constant) or w.r.t $z$ (treating $x$ and $y$ as constants).

Partial derivative of the function $f(x,y)$ w.r.t $x$ is denoted as $\frac{\partial f}{\partial x}$, $f_x$ or $f_x(x,y)$ is denoted by

$$\frac{\partial f}{\partial x}(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x},$$ for any value of $x$ and $y$ for which the limit exist. Similarly, the partial derivative (p.d) of $f(x,y)$ w.r.t $y$ written as $\frac{\partial f}{\partial y}$ or $f_y$ is defined as

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Example

Let $f(x,y) = xy^2$ using 1st principle method of differentiation, find (a) $f_x$ (b) $f_y$.

Soln.

(a) $$f_x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)y^2 - xy^2}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{xy^2 + y^2\Delta x - xy^2}{\Delta x} = \lim_{\Delta x \to 0} y^2 = y^2.$$

~~Ex-amples~~; find $f_x$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ of the fff functions.

i) $f(x,y,z) = 2x + 3y + 4z$.

ii) $f(x,y,z) = 4x^2y - 7\sqrt{y} + 3xyz^3$.

iii) $f = \frac{x}{y} + \frac{y}{z}$.

iv) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

v) $f(x,y,z) = \sin(xyz) - \frac{xy}{z}$

vi) $f(x,y,z) = \tan(x^2 - 3y^2) + 6z^3y$

vii) $f(x,y,z) = \dfrac{6x^2 - 4y + z}{\sin x + \cos y - \tan z}$

viii) $f(x,y,z) = \sqrt{2x - 4yz}\,\sec(x^2 + y^3 + z^4)$.

ix) $f(x,y) = x^3y + e^xy^2 + \ln(x^2+y^2)$.

Soln.

i) $f(x,y,z) = 4x^2y - 7\sqrt{y} + 3xyz^3$

$$f_x = 8xy + 3yz^3$$

$$\frac{\partial f}{\partial y} = 4x^2 - \frac{7}{2\sqrt{y}} + 3xz^3$$

$$\frac{\partial f}{\partial z} = 9xyz^2.$$

vi) $f(x,y,z) = \tan(x^2 - 3y^2) + 6z^3y$

$$f_x = 2x\sec^2(x^2 - 3y^2).$$

$$f_y = -6y\sec^2(x^2 - 3y^2)$$

$$f_z = 18z^2y.$$

# HIGHER ORDER PARTIAL DERIVATIVES.

Given the function $f(x,y,z)$ we can find its higher derivatives as follows; hence, the 2nd order partial derivatives are denoted by

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy}$$

**Examples;**

① Given that $f(x,y) = 6xy^3 - 4x^2y^2 + 3y$, find $f_{xx}$, $f_{yy}$, $f_{xy}$ and $f_{yx}$.

**Soln**

$f_{xx} = (f_x)_x \Rightarrow f_x = 6y^3 - 8xy^2 \Rightarrow f_{xx} = -8y^2.$

$f_{yy} = (f_y)_y \Rightarrow f_y = 18xy^2 - 8x^2y + 3 \Rightarrow f_{yy} = 36xy - 8x^2$

$f_{xy} = (f_x)_y \Rightarrow f_x = 6y^3 - 8xy^2 \Rightarrow (f_x)_y = 18y^2 - 16xy$

$f_{yx} = (f_y)_x \Rightarrow f_y = 18xy^2 - 8x^2y + 3 \Rightarrow (f_y)_x = 18y^2 - 16xy$

observe that $f_{xy} = f_{yx}$.

**Exercises;** ① find all the 2nd order partial derivatives of

ⓐ $f(x,y) = \cos(2x) - x^2e^{5y} + 3y^2$

ⓑ $f(x,y) = xe^{-x^2y^2}$

② find the indicated derivatives for each of the f/f.

(b) $f(x,y) = 3e^{xy}$ ; $\dfrac{\partial^3 f}{\partial y \, \partial x^2}$.

Example; If $z(x+y) = x^2 + y^2$; Show that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

Soln

$$z(x+y) = x^2 + y^2 \Rightarrow z = \frac{x^2+y^2}{x+y}$$

$$\frac{\partial z}{\partial x} = \frac{(x+y)2x - (x^2+y^2)\cdot 1}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x+y)\cdot 2y - (x^2+y^2)\cdot 1}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} = \frac{2x(x+y) - (x^2+y^2)}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{2y(x+y) - (x^2+y^2)}{(x+y)^2} = \frac{y^2+2xy-x^2}{(x+y)^2}.$$

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{x^2+2xy-y^2-\left[y^2+2xy-x^2\right]}{(x+y)^2}$$

$$= \frac{x^2+2xy-y^2-y^2-2xy+x^2}{(x+y)^2}$$

$$= \frac{2x^2-2y^2}{(x+y)^2} = \frac{2(x-y)(x+y)}{(x+y)^2} = \frac{2(x-y)}{x+y}$$

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \frac{4(x-y)^2}{(x+y)^2}$$

Consider the R.H.S.

$$4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]$$

$= 4\left[1 - \frac{x^2+2xy-y^2}{\ldots} - \frac{y^2+2xy-x^2}{\ldots}\right]$

$$= 4\left[\frac{(x+y)^2 - x^2 - 2xy + y^2 - y^2 - 2xy + x^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2 - 2xy + y^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{x-y}{x+y}\right]^2$$

Exercise ① If $u = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$, show that it satisfies the Laplace equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

② If $Z = \tan(3x+4y)$ find $Z_x$, $Z_y$, $Z_{xx}$, $Z_{xy}$, $Z_{yy}$

## DIFFERENTIALS.

Given the function $w = f(x,y,z)$, the differential $dw$ or $\partial f$ is given by

$$\partial w = f_x\,dx + f_y\,dy + f_z\,dz$$
$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

Example; compute the differential for each of the fff functions.

ⓐ $Z = e^{x^2+y^2}\tan(2x)$ ⓑ $f(x,y,z) = \dfrac{y^3 z^6}{x^2}$

Note; if $w = f(x_1, x_2, \ldots x_n)$; the total differential $dw$ is given by

$$dw = f_{x_1}dx_1 + f_{x_2}dx_2 + \ldots + f_{x_n}dx_n.$$

## CHAIN RULE

If $Z$ is a function of $x$ and $y$ i.e $Z = f(x,y)$ and $x, y$ are both functions of $u$ and $v$, then $Z$ is a function of $u$ and $v$. Thus

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

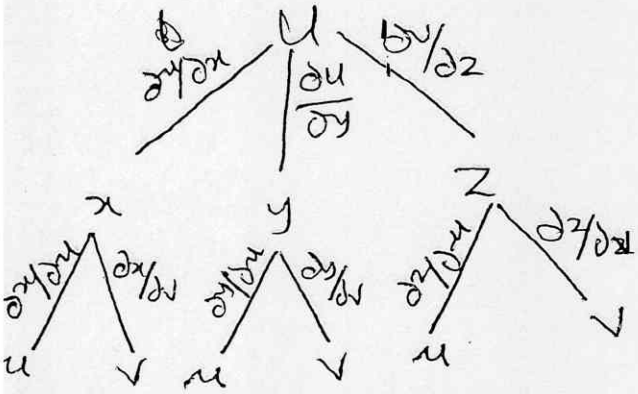
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}.$$

if $w = f(x,y)$, $x = f(t)$, $y = f(t)$, then

$\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial t}$ is called the derivative of $w$ w.r.t $t$ and if $w = f(x,y,z)$ then

$\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial t} + \dfrac{\partial f}{\partial z}\cdot\dfrac{\partial z}{\partial t}$ is called the rate of change of $f$ w.r.t $t$.

Again if $u = f(x,y,z)$, $x = \phi(u,v)$, $y = \psi(u,v)$ and $z = \theta(u,v)$.




$$\frac{\partial u}{\partial u} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$\frac{\partial u}{\partial v} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial v}$$

Example; if $u = x^2 + 3xy - y\ln z$
$x = s + t^2$; $y = s - t^2$; $z = 2t$

find (a) $\dfrac{\partial v}{\partial s}$ (b) $\dfrac{\partial v}{\partial t}$.

Soln

(a) $$\frac{\partial v}{\partial s} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial s}.$$

$\dfrac{\partial v}{\partial x} = 2x + 3y$; $\dfrac{\partial v}{\partial y} = 3x - \ln z$; $\dfrac{\partial v}{\partial z} = -y/z$

$$\frac{\partial v}{\partial s} = (2x+3y)\cdot 1 + (3x-\ln z)\cdot 1 + 0 = 5x+3y-\ln z.$$

(b) Soln:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial t}.$$

$$\frac{\partial u}{\partial x} = 2x+3y;\quad \frac{\partial v}{\partial y} = 3x-\ln z;\quad \frac{\partial u}{\partial t} = 2;\quad \frac{\partial v}{\partial z} = \frac{-y}{z}$$

$$\cancel{\frac{\partial u}{\partial t}} = (\cancel{2x}+\cancel{3y})\cdot\cancel{2t} + (3x-\cancel{\ln z})(\cancel{-2t})$$

$$\frac{\partial x}{\partial t} = 2t;\quad \frac{\partial u}{\partial t} = -2t;\quad \frac{\partial z}{\partial t} = 2.$$

$$\frac{\partial u}{\partial t} = (2x+3y)\cdot 2t + (3x-\ln z)(-2t) + \left(\frac{-y}{z}\right)2.$$

$$= 2t(2x+3y) - 2t(3x-\ln z) - \frac{2y}{z}$$

$$= 4tx + 6ty - 6tx + 2t\ln z - \frac{2y}{z}$$

$$= -2t(x+3y) + 2t\ln z - \frac{2y}{z},$$

Example; If $z = x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$.

Soln.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial x} = 2x;\quad \frac{\partial z}{\partial y} = -2y$$

$$x = r\cos\theta;\quad y = r\sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta;\quad \frac{\partial y}{\partial r} = \sin\theta.$$

$$\frac{\partial z}{\partial r} = 2x\cos\theta - 2y\sin\theta = 2(r\cos\theta\cos\theta - r\sin\theta\sin\theta)$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}.$$

But $\frac{\partial x}{\partial \theta} = -r\sin\theta$ ; $\frac{\partial y}{\partial \theta} = r\cos\theta$.

$$\frac{\partial z}{\partial \theta} = 2x(-r\sin\theta) - 2y(r\cos\theta).$$
$$= -2rx\sin\theta - 2ry\cos\theta$$
$$= -2r.r\cos\theta.\sin\theta - 2r.r\sin\theta.\cos\theta$$
$$= -2r^2\cos\theta\sin\theta - 2r^2\sin\theta\cos\theta$$
$$= -2r^2(\cos\theta\sin\theta + \sin\theta\cos\theta).$$
$$= -2r^2.2\sin\theta\cos\theta$$
$$\frac{\partial z}{\partial \theta} = -4r^2\sin\theta\cos\theta.$$

Exercises:

① If $z = x + f(u)$ where $u = xy$, show that $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = x$

② If $w = x^2 e^{2y}\cos 3z$, find $\frac{dw}{dt}$ at the point $(1, \ln 2, 0)$ on the curve $x = \cos t$, $y = \ln(t+2)$, $z = t$

③ Show that $u = e^{xy}\sin(x^2 - 2y)$ is a solution to $u_x + xu_y = (x^2+y)u$

④ find the derivative of $w = xy$ w.r.t $t$ along the path $x = \cos t$, $y = \sin t$ what is the derivative's value at $t = \pi/2$?

⑤ Given that $x(t) = 1 + at^2$ and $y(t) = bt^3$, find the rate of change of $f(x,y) = xe^{-y}$ w.r.t $t$.

Let $u = z\sin\left(\frac{y}{x}\right)$; $x = 3r^2 + 2s$, $y = 4r - s^3$, $z = 2r^2 - 3s^2$

find ① $\frac{\partial u}{\partial r}$, ② $\frac{\partial u}{\partial s}$.

(7) If $x = \rho\cos\phi$, $y = \rho\sin\phi$ Show that

$$\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial v}{\partial \phi}\right)$$

Hint $\quad V_\rho = V_x x_\rho + V_y y_\rho = \ldots\ldots \quad (1)$

$\quad V_\phi = V_x x_\rho + V_y y_\phi = \ldots\ldots \quad (2)$

Divide (2) by $\rho$ to obtain eqn (3)

Square eqn (1), eqn (3) then add them.

(8) Show that $z = f(x^2 y)$, where $f$ is differentiable

Satisfies $x\left(\frac{\partial z}{\partial x}\right) = 2y\left(\frac{\partial z}{\partial y}\right)$: Hint Let $u = x^2 y$, $z = f(u)$

) If $F(x,y) = x^4 y^2 \sin^{-1}\left(\frac{y}{x}\right)$, Show that

$$xF_x + yF_y = 6F$$

) prove that $\quad y = f(x+at) + g(x-at)$ Satisfies

$$y_{tt} = a^2 y_{xx}.$$

) If $z = x^2 \tan^{-1}\left(\frac{y}{x}\right)$, find $\frac{\partial^2 z}{\partial x \partial y}$ at $(1,1)$

) If $f(x,y,z) = x^2 e^{2y} - \frac{2}{3}y^2 z^3 + \frac{43x}{z} - 7\tan\left(\frac{7y}{2}\right)$

find $\frac{\partial^2 f}{\partial y \partial x}$, $f_{yz}$, $f_{xyz}$, $f_{yyy}$, $\frac{\partial^3 f}{\partial x \partial y \partial z}$,

$f_{xxyyzz}$, $\frac{\partial^4 f}{\partial x \partial y \partial z^2}$, $f_{xxyz}$, $\frac{\partial^5 f}{\partial x \partial z^2 \partial y^2}$.

## IMPLICIT FUNCTIONS.

The concept of partial differentiation can also be applied to find the derivative of Implicit functions. Hence, if $f(x,y) = 0$ is an Implicit function then,

$$\frac{dy}{dx} = \frac{-f_x}{f_y}$$

Examples.

## Solⁿ

Let $f(x,y) = x^2 + 4y^2 - 16 = 0$.

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 8y$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-2x}{8y} = \frac{-x}{4y}$$

**Example**; find $y'$ if $2x^2 - y^3 + 4xy - 2x = 0$

## Solⁿ.

$$f(x,y) = 2x^2 - y^3 + 4xy - 2x$$

$$f_x = 4x + 4y - 2, \qquad f_y = -3y^2 + 4x$$

at $(1,-2)$. i.e $x=1$ and $y=-2$.

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-(4x+4y-2)}{-3y^2+4x} = \frac{-(4-8-2)}{3(4)-4(1)} = \frac{-3}{4}$$

**Exercises**;

1. find all the first and Second partial derivatives of $x^3y^2 - 2x^2y + 3x$
2. If $Z = (2x-y)(x+3y)$; find $Z_x$ and $Z_y$
3. find the Slope of the tangents to the curve $y^3 + 2x^2y - 3x - 3 = 0$ at the point $(2,1)$
4. If $Z = x^4 + 2x^2y + y^3$ and $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ in the Simplest form.

## EXTREMA

The absolute maximum and absolute minimum values of a function are referred to as the extreme values of F or the local extreme of f.

i) A point on a graph at which the derivative $\left(\frac{dy}{dx} = 0\right)$

(2) A stationary point is a minimum point if $\frac{d^2f}{dx^2} > 0$

(3) A stationary point is a Maximum point if $\frac{d^2f}{dx^2} < 0$

(4) A stationary point is a point of inflexion or Saddle point if $\frac{d^2f}{dx^2} = 0$.

$\Rightarrow$ To find the minimum or Maximum point of a function of two variables $f(x, y)$

Step 1; Set $f_x$ and $f_y$ equal to Zero and Solve as a pair of Sammultineous equations for $x$ and $y$.

Step 2; If at these points, $f_{xx}$ and $f_{yy}$ are both positive, Then we have a minimum point or if $f_{xx}f_{yy} > (f_{xy})^2$

Step 3; If both $f_{xx}$ and $f_{yy}$ are negetive, we have a Maximum point.

Step 4; If $f_{xx} > 0$, $f_{yy} < 0$ or one of them is Zero, we have a Saddle point.

## FIRST DERIVATIVE TEST.

Let c be a Critical number of a function f and let (a, b) be an open Interval Containing c. the function f is differentiable on (a, b) Containing c and $f'(c) = 0$ if

i) $f(c) > 0$, Then f has a Local maxima.

ii) $f(c) < 0$, " f " " " minima.

Example; use the 1st derivatives test to find the local maxima of $f(x) = x^3 + 3x^2 - 9x + 1$

Soln

$f(x) = x^3 + 3x^2 - 9x + 1$ (1)

To find the critical point, we set $f'(x)=0$ in (ii)

$3x^2+6x-9=0 \Rightarrow x^2+2x-3=0$

$(x-1)(x+3)=0 \Rightarrow x=-3$ or $x=1$

The critical point of $f(x)$ are $-3$ and $1$.

To find the local extrema of $f$, we find $f(-3)$ and $f(1)$ using (i)

$f(x)=x^3+3x^2-9x+1$

$f(-3)=(-3)^3+3(-3)^2-9(-3)+1$

$=-27+27+27+1=28>0 \Rightarrow$ local maxima

Also,

$f(1)=1^3+3(1)^2-9(1)+1$

$=-4<0 \Rightarrow$ we have a local minima.

## Second Derivative Test

i) if $f''(c)<0$, then $f$ has local max at $c$

ii) if $f''(c)>0$ " $f$ " " min at $c$

Example; use the Second derivative test to find local extrema of $f$ if $f(x)=-2x^3+15x^2-36x+7$.

Soln

$f(x)=-2x^3+15x^2-36x+7$ —— (i)

$f'(x)=-6x^2+30x-36$ —— (ii)

$f''(x)=-12x+30$ —— (iii)

at the stationary point $f'(x)=0$

$-6x^2+30x-36=0$

$x^2-5x+6=0$

$\Rightarrow x=2$ or $x=3$.

The critical points are 2 and 3.

To get the local extrema of f, we find $f''(2)$ and $f''(3)$ using (iii)

$f''(2) = -12(2) + 30 = 6 > 0$ which has a local minimum

$f''(3) = -12(3) + 30 = -6 < 0$ " " " maximum

Example; find and classify all the critical points of (a) $f(x,y) = 4 + x^3 + y^3 - 3xy$

(b) $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

Soln

(a) $f(x,y) = 4 + x^3 + y^3 - 3xy$

$f_x = 3x^2 - 3y$ , $f_y = 3y^2 - 3x$

Setting $f_x = 0$ and $f_y = 0$ we have

$3x^2 - 3y = 0$ — ① $\cancel{\Rightarrow y(x)}$ and $3y^2 - 3x = 0$ — ②

from ② $x = y^2$

Substi $x = y^2$ in ① gives

$3y^4 - 3y = 0 \Rightarrow y(y^3 - 1) = 0$

$\Rightarrow y(y-1)(y^2 + y + 1) = 0$

$y = 0$; $y = 1$; $y = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$ ; $y = -\frac{1}{2} - \frac{\sqrt{3}i}{2}$

from $x = y^2$

$x = 0$; $x = 1$; $x = -\frac{1}{2} - \frac{\sqrt{3}i}{2}$ ; $x = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$

The critical points are

$(0,0)$, $(1,1)$, $\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}, -\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)$, $\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}, -\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)$

$f_x = 3x^2 - 3y$ $\qquad$ $f_y = 3y^2 - 3x$

$f_{xx} = 6x$ $\qquad$ $f_{yy} = 6y$

$f_{xx}(0,0) = 0$ $\qquad$ $f_{yy}(0,0) = 0$

Again :

$f_{xx}(1,1) = 6 > 0$ $\qquad$ $f_{yy}(1,1) = 6 > 0$

The critical point $(1,1)$ is a minimum point.

Solve for the ~~other~~ remaining critical points.

Theorem; Let $z = f(x,y)$ have first and second partial derivatives in an open set including a point $(x_0, y_0)$ at which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. Define $\Delta = \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)^2 \left(\frac{\partial^2 f}{\partial y^2}\right)$

Assume $\Delta < 0$ at $(x_0, y_0)$ Then,

$$z = f(x,y) \text{ has } \begin{cases} \text{a relative minimum at } (x_0,y_0) \text{ if } \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} > 0 \\ \text{a relative maximum at } (x_0,y_0) \text{ if } \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} < 0 \end{cases}$$

If $\Delta > 0$, there is neither a relative maximum nor a relative minimum at $(x_0, y_0)$.

If $\Delta = 0$, we have no Information.

<u>Example</u>: Examine $f(x,y) = x^3 + y^3 + 3xy$ for maximum and minimum values.

<u>Soln</u>

$f(x,y) = x^3 + y^3 + 3xy$

$f_x = 3(x^2 + y) = 0$ $\quad$ and $\quad$ $f_y = 3(y^2 + x) = 0$

$\Rightarrow y = -x^2$ $\qquad\qquad$ $x = -y^2$

$y = -(-y^2)^2 = -y^4$

$y + y^4 = 0 \Rightarrow y(y^3+1) = 0$

$\Rightarrow y(y+1)(y^2 - y + 1) = 0$

$\Rightarrow y = 0$ or $y = -1$

Therefore, $x = 0$ or $x = -1$

at $(0,0)$ $\frac{\partial^2 f}{\partial x^2} = 6x = 0$; $\frac{\partial^2 f}{\partial x \partial y} = 3$, and $\frac{\partial^2 f}{\partial y^2} = 6y = 0$

$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) = 9 > 0$

and $(0,0)$ yield neither a relative maximum nor minimum.

at $(-1,-1)$, $\frac{\partial^2 f}{\partial x^2} = -6$, $\frac{\partial^2 f}{\partial x \partial y} = 3$, and $\frac{\partial^2 f}{\partial y^2} = -6$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) = -27 < 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} < 0$$

Hence, $f(-1,-1) = 1$ is a relative maximum value of the function.

## LINE INTEGRAL.

The line Integral of $f(x,y)$ along $C$ is denoted by $L = \int f(x,y)\,ds$. Because of the $ds$ this is sometimes called the line Integral of $f$ with respect to arc length

$L = \int_a^b ds$ where $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ in parametric eqns.

Example

Evaluate $\int_C xy^4\,ds$ where $C$ is the right half of the circle, $x^2+y^2=16$ rotated in the counter clockwise direction

Soln

$x = r\cos t$, $y = r\sin t \implies x = 4\cos t$, $y = 4\sin t$

with $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$\frac{dx}{dt} = -4\sin t \qquad \frac{dy}{dt} = 4\cos t$

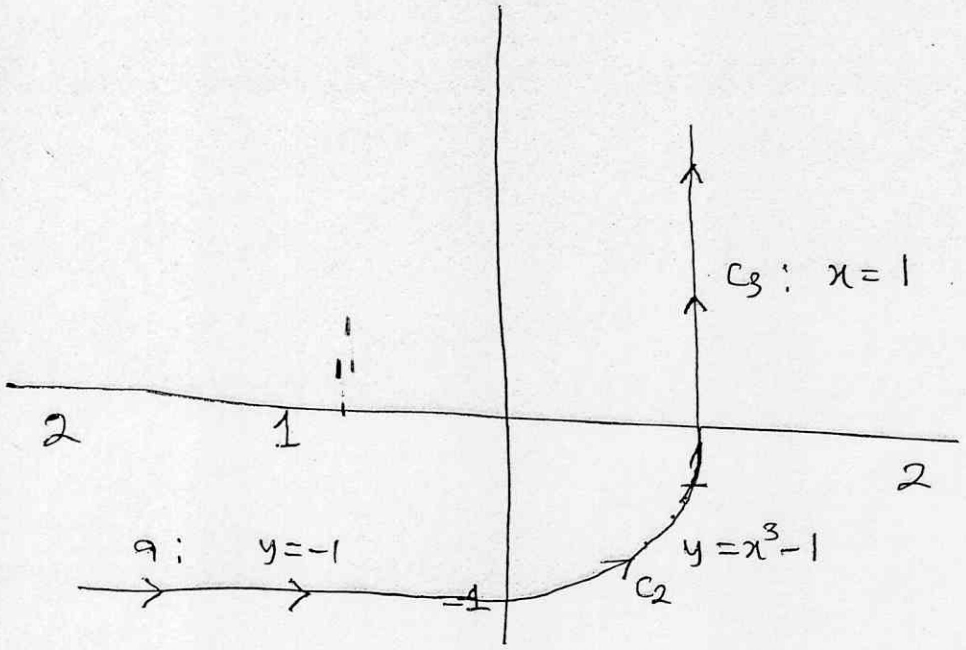
$ds = \sqrt{16\sin^2 t + 16\cos^2 t}\,dt = 4\,dt$.

$L = \int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2} 4\cos t\,(4\sin t)^4 \cdot 4\,dt$

$= 4096 \int_{-\pi/2}^{\pi/2} \cos t \sin^4 t\,dt$

$$= \frac{4096}{5} \sin^5 t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{8192}{5}.$$

<u>Example</u>: Evaluate $\int 4x^3\,ds$ where C is the curve shown below;




<u>Soln</u>

We need to parametrize each of the curves.

$C_1 : x = t, \quad y = -1; \qquad -2 \le t \le 0$

$C_2 : x = t, \quad y = t^3 - 1 \qquad 0 \le t \le 1$

$C_3 : x = 1, \quad y = t \qquad 0 \le t \le 2$

$$\int_{C_1} 4x^3\,ds = \int_{-2}^{0} 4t^3 \sqrt{1^2 + (0)^2}\,dt = \int_{-2}^{0} 4t^3\,dt = t^4 \Big|_{-2}^{0} = -16$$

$$\int_{C_2} 4x^3\,ds = \int_0^1 4t^3 \sqrt{1^2 + (3t^2)^2}\,dt = \int_0^1 4t^3 \sqrt{1 + 9t^4}\,dt$$
$$= \frac{1}{3}\left(\frac{2}{3}\right)(1 + 9t^4)^{3/2} \Big|_0^1 = 2.268$$

$$\int_{C_3} 4x^3\,ds = \int_0^2 4(1)^3 \sqrt{0^2 + 1^2}\,dt = \int_0^2 4\,dt = 8.$$

finally, $\int_C 4x^3\,ds = \int_{C_1} 4x^3\,ds + \int_{C_2} 4x^3\,ds + \int_{C_3} 4x^3\,ds$

$$= -16 + 2.268 + 8 = -5.732.$$

**Example**; Evaluate $\int_C 4x^3\,dx$ where $C$ is the line Segment from $(-2,-1)$ to $(1,2)$.

Soln

The polarization formula of the line Segment from $(-2,-1)$ to $(1,2)$ is

$$F(t) = (1-t)\langle -2,-1\rangle + t\langle 1,2\rangle = \langle -2+3t,\ -1+3t\rangle \text{ for } 0\le t\le$$

$$L = \int_C f(x,y)\,ds = \int_a^b f(x(t),\ y(t))\ \overset{\text{norm}}{\|F(t)\|}\,dt$$

$$= \int_a^b f(x(t),\ y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$L = \int_C 4x^3\,ds = \int_0^1 4(-2+3t)^3\sqrt{9+9}\,dt$$

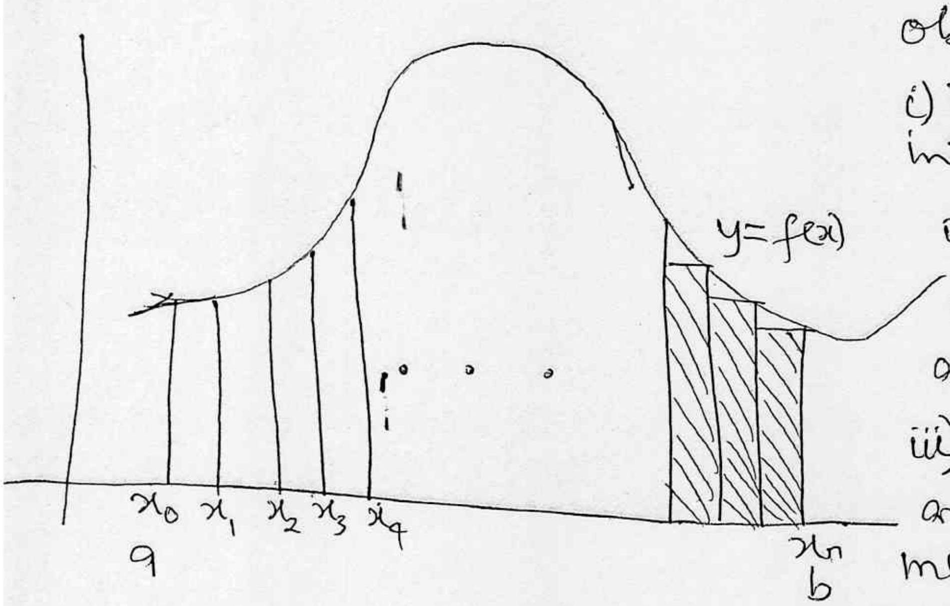
$$= 12\sqrt{2}\left(\frac{1}{12}\right)\left(-2+3t\right)^4\Big|_0^1$$

$$= 12\sqrt{2}\left(\frac{-5}{4}\right)$$

$$= -15\sqrt{2} = -21.213.$$

# MULTIPLE INTEGRAL.

So, far, we've seen that the integral of a continuous single variable function $f$ over a closed and bounded interval $[a,b]$ is the area of the region between the curve and the $x$-axis bounded left and right by $x=a$ and $x=b$ respectively. Also, this area can be approximated by taking the sum of the areas of all the rectangle that is formed under the curve as illustrated below.




Observe that

i) The interval $[a,b]$ is divided into $n$-subinterval.

ii) Area under the curve is approximately the total area of the $n$ rectangles.

iii) As $n$ gets large, the area approximation accuracy increases i.e if $S_n$ = Sum of area of $n$ rectangles.

Then $\lim\limits_{n\to\infty} S_n = \int_a^b f(x)\,dx$.

# DOUBLE INTEGRAL

The method of double integral is used to integrate a function of two variables, $f(x,y)$. Just like we integrated a single variable function over an interval, in this case we integrate $f(x,y)$ over a region $R$ of two dimensional space (i.e $\mathbb{R}^2$) and is given by

$$\iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dy\,dx$$

CASE 1; Assume the region $R$ is rectangular defined by

$$R = [a,b] \times [c,d]$$

$$= \{(x,y) : x \in [a,b] \text{ and } y \in [c,d]\} = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$

In this case, to carry out the integral $\iint_R f(x,y)\,dy\,dx$

i) Integrate w.r.t $y$ (treating $x$ as constant), then

ii) Evaluate the resulting integral from $y=c$ to $y=d$

iii) Lastly, you integrate the current result from ii) w.r.t $x$ from $x=a$ to $x=b$.

Example; Compute each of the following double integrals over the indicated rectangles.

ⓐ $\iint_R 6xy^2\,dA$, $R=[2,4]\times[1,2]$ ⓑ $\iint_R x^2y^2+\cos(\pi x)+\sin(\pi y)\,dA$, $R=[-2,-1]\times[0,1]$.

ⓑ $\iint_R (2x-4y^3)\,dA$, $R=[-5,4]\times[0,3]$.

ⓒ $\iint_R \frac{1}{(2x+3y)^2}\,dA$, $R=[0,1]\times[1,2]$ ⓔ $\iint_R xe^{xy}\,dA$, $R=[-1,2]\times[0,1]$.

Soln

ⓐ $$\iint_R 6xy^2\,dA = \int_2^4\int_1^2 6xy^2\,dy\,dx$$

$$= \int_2^4 \left[6x\frac{y^3}{3}\right]_1^2 dx = 2\int_2^4 (8x-x)\,dx$$

$$= 2\int_2^4 7x\,dx$$

$$= 14\left[\frac{x^2}{2}\right]_2^4$$

$$= 84.$$

# Properties of Double Integral

1) $\iint\limits_R f(x,y) \pm g(x,y)\,dA = \iint\limits_R f(x,y)\,dA \pm \iint\limits_R g(x,y)\,dA$

2) $\iint\limits_R c f(x,y)\,dA = c\iint\limits_R f(x,y)\,dA$, $c$ is Constant.

3) If the region $R$ Can be Split into Separate region $R_1$ and $R_2$ Respectively, Then

$$\iint\limits_R f\,dA = \iint\limits_{R_1} f\,dA + \iint\limits_{R_2} f\,dA.$$

## FUBINI'S THEOREM.

If $f(x,y)$ is Continuous throughout the rectangular region, $R: a \le x \le b$, $c \le y \le d$, Then

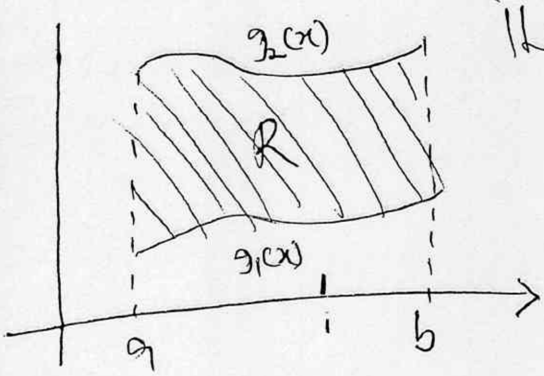
$$\cancel{\iint\limits_R f(x,y)\,dA = \int_a^b\int_c^d}$$

$$\iint\limits_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx$$

CASE 2; Assume the region $R$ is non-rectangular Then we have the following possibilities;

i) If $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$

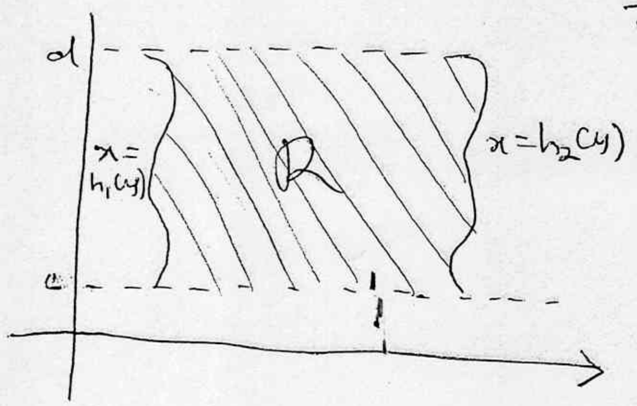
That is




Then $\iint\limits_R f(x,y)\,dA = \int_a^b\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx$

where $g_1$ and $g_2$ are cont on $[a,b]$

ii) If $h_1(y) \le x \le h_2(y)$ and $c \le y \le d$
that is




Then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\,dy$$

where $h_1$ and $h_2$ are continuous on $[c, d]$.

Example; Evaluate the following Integrals over the indicated regions.

i) $\iint_R 2y^3 e^{xy}\,dA$, $R = \{(x,y) : 0 \le x \le y^2,\ 0 \le y \le 1\}$

ii) $\iint_R \frac{3e^{y/\sqrt{x}}}{2}\,dA$; $R = \{(x,y) : 1 \le x \le 4,\ 0 \le y \le \sqrt{x}\}$

iii) $\iint_R 4xy - y^3\,dA$; $R = \{(x,y) : 0 \le x \le 1,\ x^3 \le y \le \sqrt{x}\}$

iv) $\iint_R \frac{1}{x}\cos\frac{y}{x}\,dA$; $R = \{\frac{\pi}{2} \le x \le \pi;\ 0 \le y \le x^2\}$.

v) $\int_0^1 \int_0^{\pi/2 - y} \cos(x+y)\,dx\,dy$.

Soln

i) $\int_0^1 \int_0^{y^2} 2y^3 e^{xy}\,dx\,dy = 2\int_0^1 \left(\int_0^{y^2} y^3 e^{xy}\,dx\right)dy$

$= 2\int_0^1 \left( y^3 \left(\frac{e^{xy}}{y}\right)\Big|_0^{y^2} \right) dy = 2\int_0^1 \frac{y^3}{y}\left(e^{xy}\right)\Big|_{x=0}^{x=y^2} dy$

$= 2\int_0^1 y^2\left(e^{y^3} - 1\right) dy = 2\int_0^1 y^2 e^{y^3}\,dy - 2\int_0^1 y^2\,dy$

$= 2\left[\int_0^1 y^2 e^u \cdot \frac{du}{3y^2}\,du\right] - 2\left[\frac{y^3}{3}\right]_0^1$

$= \frac{2}{3}\int_0^1 e^u\,du - \frac{2}{3}\left[y^3\right]_0^1$

$$= \frac{2}{3}\left[e^{y^3}\right]_0^1 - \frac{2}{3}\left[y^3\right]_0^1$$

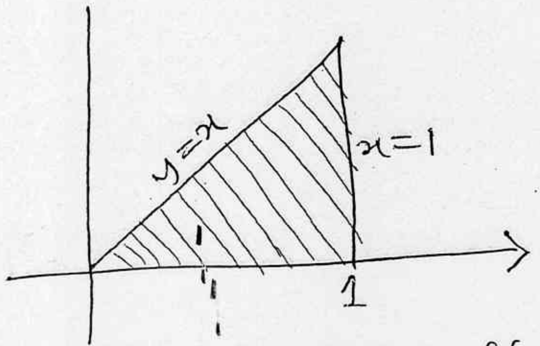
$$= \frac{2}{3}\left[e-1\right] - \frac{2}{3}(1) = \frac{2}{3}e - \frac{4}{3}.$$

Example; Evaluate

$\iint_\Delta (3-x-y)\,dA$, where $\Delta$ is the triangle in the $(x,y)$ plane bounded by the $x$-axis and the lines $y=x$ and $x=1$

Soln

(A)




Using (A) above, to find $x$ and $y$ limits of integration, observe that, the line L enters the region at $x=y$ and leaves at $x=1$ while $0 \le y \le x$ through out.

Hence the region becomes $0 \le x \le 1$ and $0 \le y \le x$

we now evaluate.

$$\int_0^1 \int_y^1 (3-x-y)\,dx\,dy = \quad \cdots \quad = 1$$

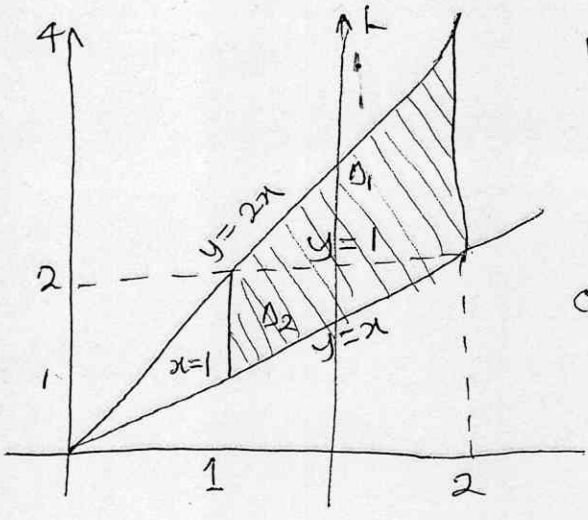
Example; i) Integrate $f(x,y) = \frac{x}{y}$ over the region in the 1st quadrant, bounded by the lines $y=x$, $y=2x$, $x=1$ and $x=2$

ii) $f(x,y) = xy$ over the region bounded by $y=x$, $y=x$ and $x+y=2$

Soln

## Soln

i) $f(x,y) = \frac{x}{y}$ ; $y = x$, $y = 2x$, $x = 1$, $x = 2$




Using line L; let $D = D_1 \cup D_2$

$D_1$: $1 \le x \le 2$; $2 \le y \le 2x$

$D_2$: $1 \le x \le 2$; $x \le y \le 2$

or, take two parallel lines to $x$-axis

$D_1$: $y/2 \le x \le 2$ ; $2 \le y \le 4$

$D_2$: $1 \le x \le y$ ; $1 \le y \le 2$

$$\iint_D \frac{x}{y}\, dA = \int_1^2 \int_2^{2x} \frac{x}{y}\, dy\, dx + \int_1^2 \int_x^2 \frac{x}{y}\, dy\, dx =$$

**Example**; Evaluate $\iint_D (6x^3 - 40y)\, dA$ over the region bounded by the triangle with vertices $(0,3)$, $(1,1)$ and $(5,3)$

$D_1 = \{(x,y) \mid 0 \le x \le 1, \ -2x+3 \le y \le 3\}$; $D_2 = \{(x,y) \mid 1 \le x \le 5, \ \frac{1}{2}x + \frac{1}{2} \le y \le 3\}$

## Soln

$$\iint (6x^2 - 40y)\, dA = \iint_{D_1} 6x^2 - 40y\, dA + \iint_{D_2} 6x^2 - 40y\, dA$$

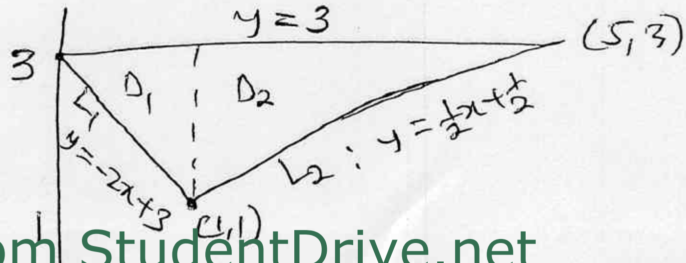
$D_1$; $y - 3 = -2(x-0)$

$\Rightarrow y = -2x+3$

$D_2$; $y - 1 = \frac{3-1}{5-1}(x-1)$

$\Rightarrow y = \frac{1}{2}x + \frac{1}{2}$.

$$= \int_0^1 \int_{-2x+3}^3 (6x^2 - 40y)\, dy\, dx + \int_1^5 \int_{\frac{1}{2}x+\frac{1}{2}}^3 (6x^2 - 40y)\, dy\, dx$$



## Exercises ;

(1) Integrate $f$ over the given region
$f(x,y) = x^2 + y^2$ over the triangular region with vertices $(0,0)$, $(0,1)$ and $(1,0)$

(2) Evaluate $\iint_D e^y \ln x \, dA$; $D$ is the region in the 1st quadrant bounded by the curve $y = \ln x$ from $x=1$ to $x=2$

(3) Evaluate $\iint_D (xy - y^3)\, dA$; where $D$ is the region ~~Const~~ Consisting the Square $\{(x,y) \mid -1 \le x \le 0,\ 0 \le y \le 1\}$ together with the triangle $\{(x,y) : 0 \le x \le 1,\ x \le y \le 1\}$.

(4) Integrate $f(x,y) = 6x^2 - 40y$ over the region bounded by the triangle with vertices $(0,3)$, $(1,1)$ and $(5,3)$ —

(5) Evaluate the f/f Integrals by reversing the order of integration.

i) $\int_0^3 \int_{x^2}^9 x^3 e^{y^3}\, dy\, dx$  ii) $\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\, dx\, dy$

iii) $\int_0^2 \int_{y^3}^{4\sqrt{2y}} (x^2 y - xy^2)\, dx\, dy$  iv) $\int_0^2 \int_{x^2}^{2x} (4x+2)\, dy\, dx$

v) $\int_0^2 \int_0^{4-x^2} \frac{\cancel{x}\, e^{2y}}{4-y}\, dy\, dx$  vi) $\int_0^{\ln 2} \int_{e^x}^2 dx\, dy$

vii) $\int_0^1 \int_{1-x}^{1-x^2} dy\, dx$  viii) $\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$

ix) $\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 6y\, dy\, dx$.

Solve the problems on reversing order of Integration

(1) $\int_0^2 \int_0^{4-y^2} y\, dx\, dy$  (2) $\int_0^2 \int_{x^2}^{2x} (4x+2)\, dx\, dy$

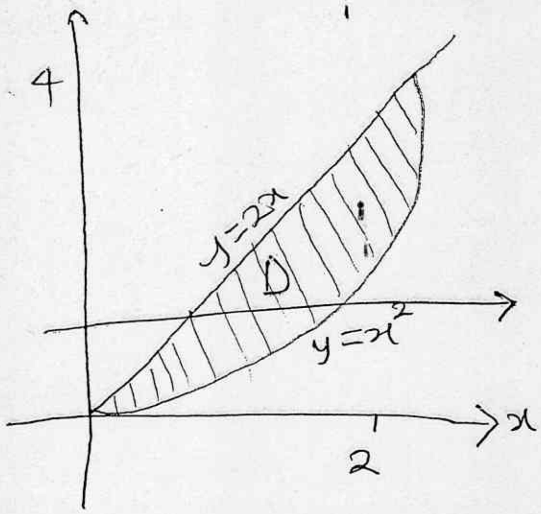
Soln

# Soln

(2) clearly $0 \le x \le 2$ $\quad x^2 \le y \le 2x$

$x^2 = 2x \Rightarrow x = 0, 2$

$\Rightarrow y = 0, 4 \Rightarrow (0,0), (2,4)$




The line enters at $x = y/2$ and leaves at $x = \sqrt{y}$

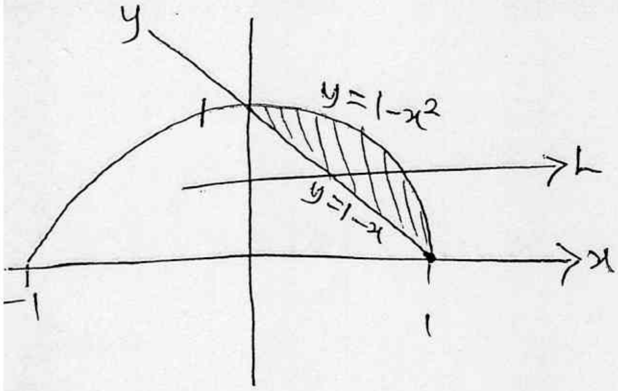
$$\iint_D (4x+2)\,dy\,dx = \int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,dx\,dy$$

Compute.

Example evaluate $\int_0^1 \int_{1-x}^{1-x^2} dy\,dx$

# Soln

clearly $0 \le x \le 1$ and $1 - x \le y \le 1 - x^2$




L enters at $1-y$ and leaves at $x = \sqrt{1-y}$ and $0 \le y \le 1$

verify

$$\int_0^1 \int_{1-x}^{1-x^2} dy\,dx = \int_0^1 \int_{1-y}^{\sqrt{1-y}} dx\,dy$$

Compute.

# APPLICATION OF DOUBLE INTEGRAL TO AREA/VOLUME

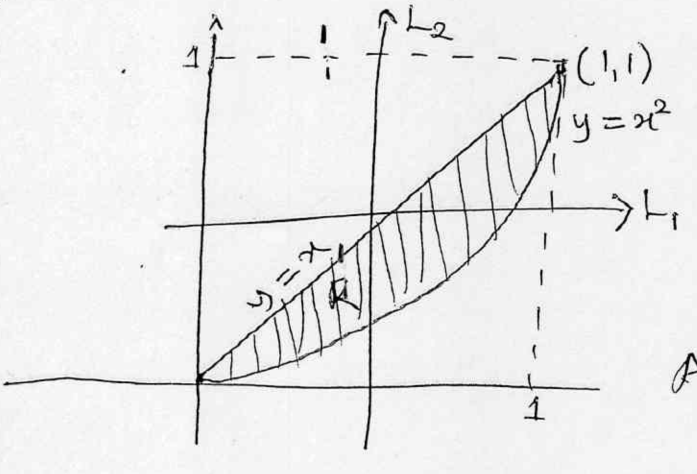
We all know that the integral of a single variable function over an interval is the area under the curve. The area of a region in $x, y$ plane can also be evaluated using the double integral given by

$$A = \iint_R dy\,dx \quad \text{or} \quad \iint_R dx\,dy;$$ where $R$ is the region of integration.

However, if the region is bounded on the left by $x = g_1(y)$ and right by $x = g_2(y)$, also bounded below by the line $y = c$ and above $y = d$, then you first integrate along $x$-axis before integrating the resulting integral along $y$-axis. That is, suppose $g_1(y) \le x \le g_2(y)$ and $c \le y \le d$ then

$$A = \int_c^d \int_{g_1(y)}^{g_2(y)} dx\,dy$$

**Example:** Calculate the area of the region bounded by $y = x^2$ and the line $y = x$ in the first quadrant.




$x^2 = x \Rightarrow x = 0, 1$

$0 \le x \le 1 \qquad y \le x \le \sqrt{y}$

if $x = 0, \quad y = 0$

$x = 1, \quad y = 1$

$$A = \int_0^1 \int_y^{\sqrt{y}} dx\,dy = \frac{1}{6}$$

or

$$A = \int_0^1 \int_{x^2}^{x} dy\,dx = \frac{1}{6} \text{ Sq unit.}$$

**Exercises:** Calculate the area of the f/f region in the $x,y$-plane, sketch the region and label each bounding curve with its equation.

i) $\int_0^6 \int_{y^2/3}^{2y} dx\,dy$ ; ii) $\int_0^3 \int_{-x}^{x(2-x)} dy\,dx$ iii) $\int_{-1}^{2} \int_{y^2}^{y+2} dx\,dy$ .

**Note:**

Evaluating double integral over a region is the volume of the solid shape under the given surface $z = f(x,y)$. It is given by

$$V = \iint_R f(x,y)\,dy\,dx \quad \text{or} \quad \iint_R f(x,y)\,dy\,dx$$

**Example:** find the volume of the solid shape that lies below the surface $z = 16xy + 200$ and above the region in the $x,y$ plane bounded by the parabola $y = x^2$ and $y = 8 - x^2$.

**Soln**

$f(x,y) = 16xy + 200$

The parabola meet at

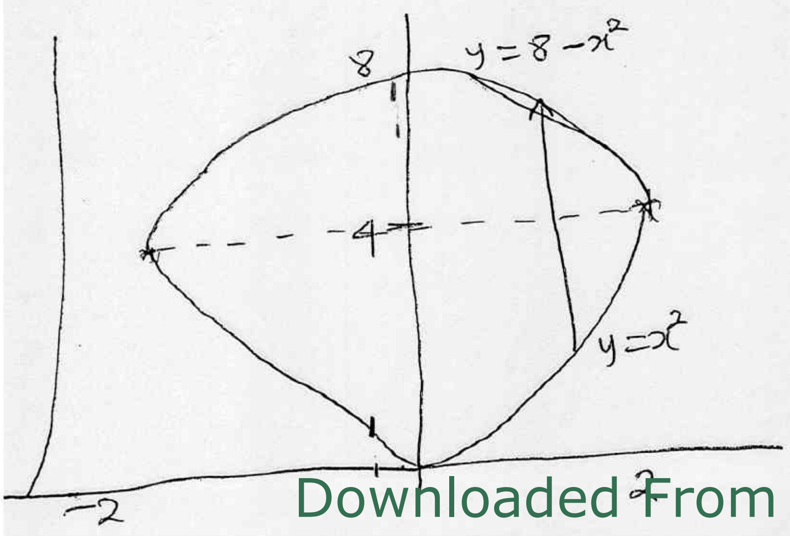
$x^2 = 8 - x^2$

$x = \pm 2$

$y = 4 \Rightarrow (-2,4), (2,4)$

$$\iint 16xy + 200\,dA$$

$$= \int_{-2}^{2} \int_{x^2}^{8-x^2} 16xy + 200\,dy\,dx$$




$$= \frac{12800}{3} \text{ cubic unit.}$$

# TRIPPLE INTEGRALS

Triple Integral is used to Integrate over a 3-dimensional region. the notation for the general triple Integral is

$$\iiint_B f(x,y,z)\,dv \quad \text{where} \quad B=[a,b]\times[c,d]\times[r,s]$$

**Example;** Evaluate the f/f Integral.

$$\iiint_B 8xyz\,dv; \quad B=[2,3]\times[1,2]\times[0,1].$$

**Soln**

$$\iiint_B 8xyz\,dv = \int_1^2\int_2^3\int_0^1 8xyz\,dz\,dx\,dy$$

$$= \int_1^2\int_2^3\left[8xy\frac{z^2}{2}\right]_0^1 dx\,dy$$

$$= \int_1^2\int_2^3 (4xy]\,dx\,dy = \int_1^2\left[2x^2y\right]_2^3 dy$$

$$= \int_1^2\left[2\cdot 9\cdot y - 2\cdot 4\cdot y\right]dy = \int_1^2 10y\,dy = 15.$$

**Exercise;** Evaluate the f/f triple Integral.

i) $\iiint_B x+y+z\,dv$, $\quad B=[-1,0]\times[1,2]\times[-3,1]$

ii) $\iiint_B xy^2z^2\,dv$, $\quad B=[0,1]\times[1,2]\times[5,6]$.

i) $\iiint_B x^2+\frac{yz}{4}\,dv$, $\quad B=[-2,-1]\times[2,3]\times[0,4]$.

# Exercises

(1) Use $\boxed{\dfrac{d(\ln u)}{dx} = \dfrac{1}{u}\dfrac{du}{dx}}$ to differentiate the f/f.

(a) $\ln 3^x x$ (b) $(\ln x)^3$ (c) $\sin^{-1}(\ln 2x)$ (d) $\ln(x^2+1)$ (e) $\tan^{-1}(\ln x)$

(f) $\ln(\tan x + \sec x)$ (g) $\ln(\cos x)^{\sec x}$

(2) Differentiate

(i) $y^2 = x(x+1)$ (ii) $y^6 = \sqrt{\dfrac{(x+1)^6}{(x+2)^2}}$ (iii) $y^{4/5} = \dfrac{\sqrt{\sin x \cos x}}{1+2\ln x}$

(iv) $y = x^{-x}$ (v) $\sqrt{y} = \dfrac{x^5 \tan^{-1}(2x)}{(3-2x)\sqrt[3]{x}}$ (vi) $x^{\sin x}$

(vii) $(\cos x)^{2x}$ (viii) $x^{\ln x}$ (ix) $\ln(x^x)$

(3) Differentiate

(a) $y = \dfrac{\cot 2x \times \sqrt{\cos x} - e^{5x}}{\sqrt{6x^2-5}}$ (b) $y = \dfrac{1}{\sqrt{x^2+1}\,\sqrt[3]{x^2-1}}$

(c) $y = \dfrac{x^2 e^x}{(x-1)^3}$ (d) $y = \dfrac{e^{x/2}\sin x}{x^4}$ (e) $y = \dfrac{x^2 e^x}{(x-1)^3}$

(f) $y = \dfrac{(x-1)^2 e^{2x}}{(2x-1)^3}$ (g) $y = x\sin^3 x \cos^2 3x$ (h) $y = \sqrt[7]{\dfrac{2x+2}{5x-1}}$

(i) $y = \dfrac{\cot^2 x\sqrt{\sec^2 x}\, e^{(3x^2-1)}}{\ln x\sqrt{1-4x^2}}$

(4) Use $\boxed{\dfrac{d}{dx}(e^u) = e^u\dfrac{du}{dx}}$ to diff the f/f.

(i) $y = e^{-\frac{1}{x}}$ (ii) $y = \sec^{-1}(e^{3x})$ (iii) $y = \ln\left(\dfrac{e^x}{1+e^x}\right)$ (iv) $y = x^2 e^2$

(v) $y = e^{\cos 2x}$ (vi) $y = \tan^{-1}(e^{2x})$ (vii) $y = e^{\sin^{-1}x}$ (viii) $y = e^{2(2\cos x - 3\sin x)}$

(5) Differentiate the f/f

(a) $y = \left(\dfrac{1}{3}\right)^x$ (b) $y = 10^{5x}$ (c) $y = 10^{x^2}$ (d) $y = \dfrac{1}{5^x}$ (e) $y = 4^{x+1}$ (f) $y = 3^{5x}$

(6) Differentiate the f/f $\tan^{-1}\sqrt{x+1}$ (iv) ~~$y =$~~

$\csc^{-1}(2x)$ (iii)

(a) $\frac{d}{du}(\sin^{-1} 2x)$ (b) $\frac{d}{du}(\sin^{-1} x^2)$

vi) if $y = e^{2x} \cos 3x$ Show that

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0.$$

) use Leibinetz formula to find the nth derivatives of $y = x^2 \log^3$

) if $y = e^{m\cos^{-1}x}$ prove that (a) $(1-x^2)y'' - xy' = m^2 y$

(b) $(1-x^2)y_{n+2} - (2n+1)y_{n+1} - (n^2+m^2)y_n = 0$ also find $y_n(0)$

) if $y = x^n \log x$ Show that $y^{n+1} = \frac{n!}{x}$

) $y^{(n)} = D^n(x^n \log x)$ prove that $y^n = ny^{(n-1)} + (n-1)!$

) if $y = a\cos(\log x) + b\sin(\log x)$. prove that

$$x^2 y^{n+2} + (2n+1)x y^{n+1} + (n^2+1)y^n = 0$$

) find nth derivative of $y = \frac{x+2}{x+1} + \log\frac{x+2}{x+1}$

) If $y = \sin(\sin x)$ prove That $y'' + \tan x\, y' + y\cos^2 x = 0$

) if $y = \frac{ax+b}{cx+d}$ Show That $2y'y''' = 3y''$

) if $y = \frac{a\sin x + b\cos x}{a\cos x - b\sin x}$ Show That $y' = 1 + y^2$.