LINEAR ALGEBRA II

VECTOR SPACE OVER THE REAL FIELD

FIELD: Let $\mathbb{K}$ be a non-empty set, then, the set $\mathbb{K}$ is said to be a field if all the following conditions are satisfied.

FL1: Closure:

a. Additive $\forall \alpha, \beta \in \mathbb{K}$, $\alpha + \beta \in \mathbb{K}$

b. Multiplicative $\forall \alpha, \beta \in \mathbb{K}$, $\alpha \cdot \beta \in \mathbb{K}$

FL2: Cummutativity:

a. Additive: $\forall \alpha, \beta \in \mathbb{K}$, $\alpha + \beta = \beta + \alpha$

b. Multiplicative: $\forall \alpha, \beta \in \mathbb{K}$, $\alpha\beta = \beta\alpha$

FL3: Associativity:

a. Addition: $\forall \alpha, \beta, \gamma \in \mathbb{K}$, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

b. Multiplicative: $\forall \alpha, \beta, \gamma \in \mathbb{K}$, $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$

FL4: Existence of Identity:

a. Additive: $\forall \alpha \in \mathbb{K}\ \exists\ 0 \in \mathbb{K}$: $\alpha + 0 = 0 + \alpha = \alpha$

b. Multiplicative: $\forall \alpha \in \mathbb{K}\ \exists\ 1 \in \mathbb{K}$: $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$

FL5: Existence of Inverse:

a. Additive: $\forall \alpha \in \mathbb{K}\ \exists -\alpha \in \mathbb{K}$: $\alpha + (-\alpha) = 0$

b. Multiplicative: $\forall \alpha \in \mathbb{K}\ \exists\ \alpha^{-1} \in \mathbb{K}$: $\alpha \cdot \alpha^{-1} = 1$

FL6: Distributivity:

a. Left: $\forall \alpha, \beta, \gamma \in \mathbb{K}$, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

b. Right: $\forall \alpha, \beta, \gamma \in \mathbb{K}$ $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$

NOTE: The element of the field $\mathbb{K}$ are called scalers

Example 1: Show that ~~each of the following sets~~ the set of real numbers together with the two binary operation (i.e Additive and Multiplicative) is a field.

~~a. Set of real numbers~~

~~b. Set of Complex numbers. $\mathbb{C} = \{(x + iy), x, y \in \mathbb{R}, i = \sqrt{-1}\}$~~

~~c. Set of Rational Numbers $\mathbb{Q} = \{q : q = a/b, a, b \in \mathbb{Z} \text{ and } b \neq 0\}$~~

Solution

$\mathbb{R} = \{x : x \in \mathbb{R}\} = (-\infty, \infty)$

FL1: Closure

a. $\forall \alpha, \beta \in \mathbb{R}, \quad \alpha + \beta \in \mathbb{R}$

b. $\forall \alpha, \beta \in \mathbb{R}, \quad \alpha \cdot \beta \in \mathbb{R}$

FL2: Commutativity

a. $\forall \alpha, \beta \in \mathbb{R}, \quad \alpha + \beta = \beta + \alpha$

b. $\forall \alpha, \beta \in \mathbb{R}, \quad \alpha \cdot \beta = \beta \cdot \alpha$

FL3: Associativity

a. $\forall \alpha, \beta, \gamma \in \mathbb{R}, \quad \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

b. $\forall \alpha, \beta, \gamma \in \mathbb{R}, \quad \alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$

FL4: Existence of Identities:

a. $\forall \alpha \in \mathbb{R}, \exists\, 0 \in \mathbb{R} \ni 0 + \alpha = \alpha + 0 = \alpha \in \mathbb{R}$

b. $\forall \alpha \in \mathbb{R}, \exists\, 1 \in \mathbb{R} \ni \alpha \cdot 1 = 1 \cdot \alpha = \alpha \in \mathbb{R}$

FL5: Existence of Inverse:

a. $\forall \alpha \in \mathbb{R}, \exists -\alpha \in \mathbb{R} \ni \alpha + (-\alpha) = \alpha - \alpha = 0$

b. $\forall \alpha \neq 0 \in \mathbb{R}, \exists\, \alpha^{-1} \in \mathbb{R} \ni \alpha \cdot \alpha^{-1} = 1$

FL6: Distributivity:

a. $\forall\ \alpha, \beta, \gamma \in \mathbb{R},\quad \alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$

b. $\forall\ \alpha, \beta, \gamma \in \mathbb{R},\quad (\alpha+\beta)\gamma = \alpha\gamma + \beta\gamma$

Hence $\mathbb{R}$ is a field.

NOTE: $\mathbb{K}$ could be any of $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$. In this course, we are to deal with $\mathbb{K} = \mathbb{R}$.

VECTOR SPACE.

DEFINATION: Let V be a non-empty set and $\mathbb{R}$ be a field of real numbers, then a vector space over the real field $\mathbb{R}$ is a set of objects in which two operation, addition '+' and scalar multiplication '.' are defined. Such that, the following conditions are satisfied

A_1: (Closure of Addition)

$\forall\ u, v \in V$, $u+v$ is defined and $u+v \in V$

A_2: (Commutativity for Addition)

$\forall\ u, v \in V,\quad u+v = v+u$

A_3: (Associativity of Addition)

$\forall\ u, v, w \in V,\ u+(v+w) = (u+v)+w$

A_4: (Existence of Additive Identity)

$\forall\ u \in V,\ \exists\ \vec{0} \in V \ni u+0 = 0+u = u$

A_5: (Existence of Additive Inverse)

$\forall\ u \in V, \exists\ -u \in V \ni u+(-u) = u-u = 0$

M_1: (Closure for Scalar Multiplication)

$\forall\, u \in W$, and $\alpha \in \mathbb{R}$, $\alpha \cdot u$ is defined and $\alpha \cdot u \in W$

M_2: (Multiplication by 1)

$\forall\, u \in W \;\exists\; 1 \in \mathbb{R} \ni 1 \cdot u = u$

M_3: (Associativity of Multiplication)

$\forall\, u \in W$, $\forall\, \alpha, \beta \in \mathbb{R}$, $\alpha \cdot (\beta \cdot u) = (\alpha \cdot \beta) \cdot u$

D_1: (first Distributive property)

$\forall\, u, v \in W$, $\forall\, \alpha \in \mathbb{R}$, $\alpha \cdot (u+v) = \alpha \cdot u + \alpha \cdot v$

D_2: (Second Distributive property)

$\forall\, u \in W$, and $\forall\, \alpha, \beta \in \mathbb{R}$, $(\alpha+\beta) \cdot u = \alpha \cdot u + \beta \cdot u$

NOTE: ✓ The element of W are Called vectors and the element of $\mathbb{R}$ are Called Scalars.

✓ Note that, $\forall\, u, v \in W$, $u+v \in W$ means W is Closed Under '+' operation and also $\forall\, u \in W$, $\forall\, \alpha \in \mathbb{R}$, $\alpha \cdot u \in W$ means W is closed under "·" operation

– Note that, since $K = \mathbb{R}$, then W is said to be Vector space over the field of real numbers.

Example 1. Consider $\mathbb{R}^2$, i.e order pairs of real 2 (tuples). Show that $\mathbb{R}^2$ is a vector space over $\mathbb{R}$.

Proof:

To proof $\mathbb{R}^2$ is a vector space, the ~~above~~ conditions mentioned above need to be satisfied.

Now, let $U = (a_1, a_2)$, $V = (b_1, b_2)$, $W = (c_1, c_2)$

want $U+V \in \mathbb{R}^2$,

A_1: $\forall u, v \in \mathbb{R}^2$, $u+v = (a_1+b_1, a_2+b_2) \in \mathbb{R}^2$ is defined and closed under addition.

A_2: $\forall u, v \in \mathbb{R}^2$, $u+v = v+u$

$u+v = (a_1+b_1, a_2+b_2)$, $v+u = (b_1+a_1, b_2+a_2)$

but addition is commutative.

$\therefore u+v = v+u$

A_3 $\forall u, v, w \in \mathbb{R}^2$, $u+(v+w) = (u+v)+w$

$u+(v+w) = (a_1+b_1+c_1, a_2+b_2+c_2)$

$(u+v)+w = (a_1+b_1+c_1, a_2+b_2+c_2)$

$\therefore u+(v+w) = (u+v)+w$

A_4: $\forall u \in \mathbb{R}^2$, $\exists \vec{0} \in \mathbb{R}^2 \ni u+\vec{0} = u$

$\vec{0} = (0, 0)$, $u+\vec{0} = (a_1+0, a_2+0) = (a_1, a_2)$

$\therefore u+\vec{0} = u$

A_5: $\forall u \in \mathbb{R}^2 \exists -u \in \mathbb{R}^2 \ni u+(-u) = 0$

$-u = (-a_1, -a_2)$

$u+(-u) = (a_1, a_2) + (-a_1, -a_2)$

$= (a_1 - a_1; a_2 - a_2) = (0, 0)$

$\therefore u+(-u) = \vec{0}$

M_1: $\forall\, u \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, want $\alpha u \in \mathbb{R}^2$

$\alpha \cdot u = \alpha(a_1, a_2) = (\alpha a_1, \alpha a_2) \in \mathbb{R}^2$ is defined and closed under multiplication

M_2: $\forall\, u \in \mathbb{R}^2 \ \exists\ 1 \in \mathbb{R} \ni 1 \cdot u = u$ or $u \cdot 1 = u$

$u \cdot 1 = (a_1 \cdot 1, a_2 \cdot 1) = (a_1, a_2)$

$\therefore u \cdot 1 = u$

M_3: $\forall\, u \in \mathbb{R}^2 \ \exists\ \alpha, \beta \in \mathbb{R} \ni \alpha(\beta u) = (\alpha\beta) u$

$\alpha(\beta u) = \alpha(\beta a_1, \beta a_2) = \alpha\beta(a_1, a_2)$

$\therefore \alpha(\beta u) = (\alpha\beta) u$

D_1: $\forall\, u, v \in \mathbb{R}^2 \ \exists\ \alpha \in \mathbb{R} \ni \alpha(u+v) = \alpha u + \alpha v$

$\alpha(u+v) = \alpha(a_1 + b_1, a_2 + b_2)$

$= \alpha(a_1, a_2) + \alpha(b_1, b_2)$

$\therefore \alpha(u+v) = \alpha u + \alpha v$

D_2: $\forall\, u \in \mathbb{R}^2, \ \exists\ \alpha, \beta \in \mathbb{R} \ni \alpha(\beta \cdot u) = (\alpha\beta) u$

$\alpha(\beta u) = \alpha(\beta a_1, \beta a_2) = \alpha\beta(a_1, a_2)$

$\therefore \alpha(\beta u) = (\alpha\beta) u$.

Hence $\mathbb{R}^2$ is a vector space over $\mathbb{R}$.