Euclidean Space ($\mathbb{R}^n$)

A Euclidean space $\mathbb{R}^n$ is the set of all n-dimensional vectors (n-tuples) of the form $(a_1, a_2, \ldots, a_n)$ where $a_1, a_2, \ldots, a_n$ are real numbers called the components of the vectors $\bar{x}$. The operation of addition and multiplication can be extended to vectors in $\mathbb{R}^n$.

For instance, Let

$$\bar{x} = (x_1, x_2, \ldots, x_n) \text{ and } \bar{y} = (y_1, y_2, \ldots, y_n)$$

Then

$$\bar{x} + \bar{y} = (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n)$$
$$= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

Also,

$$\lambda \bar{x} = \lambda (x_1, x_2, \ldots, x_n)$$
$$= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n) \; ; \; \lambda \in \mathbb{R}.$$

Vector Spaces

Defn:- A real vector space is a set V of elements with two operations of addition $\oplus$ and scalar multiplication $\odot$ defined on it and satisfying the following axioms.

A_1 : $\forall \bar{x}, \bar{y} \in V, \; \bar{x} + \bar{y} \in V$ (closure properties).

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2

A_2: $\bar{x} + \bar{y} = \bar{y} + \bar{x}$, $\forall \bar{x}, \bar{y} \in V$ (Commutative)

A_3: $\bar{x} + (\bar{y} + \bar{z}) = (\bar{x} + \bar{y}) + \bar{z}$, $\forall \bar{x}, \bar{y}, \bar{z} \in V$ (Associative)

A_4: For any $\bar{x} \in V$, $\exists\ \bar{0} \in V$ such that

$$\bar{x} + \bar{0} = \bar{x} \quad \text{(additive identity)}.$$

A_5: For each $\bar{x} \in V\ \exists\ -\bar{x} \in V \ni$

$$\bar{x} + (-\bar{x}) = 0 \quad \text{(additive inverse)}.$$

M_1: For any $\bar{x} \in V$ and any scalar λ, then

$$\lambda . \bar{x} \in V \quad \text{(closure under Multiplication)}$$

M_2: $\lambda(\mu \bar{x}) = (\lambda \mu)\bar{x}$, $\forall \lambda, \mu \in \mathbb{R}$, $\bar{x} \in V$. (Scalar Associativity)

M_3: $1 \cdot \bar{x} = \bar{x} \cdot 1 = \bar{x}$, $\forall \bar{x} \in V$ (Multiplicative Identity)

D_1: $\lambda(\bar{x} + \bar{y}) = \lambda\bar{x} + \lambda\bar{y}$ $\forall \bar{x}, \bar{y} \in V$, $\lambda \in \mathbb{R}$

(Left distributive over Vectors addition)

D_2: $(\lambda + \mu)\bar{x} = \lambda\bar{x} + \mu\bar{x}$ $\forall \lambda, \mu \in \mathbb{R}$, $\bar{x} \in V$

(Right Distributive over scalar addition).

NB: The elements of a vector space are usually called Vectors.

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Examples

1. Let $V = \mathbb{R}^2 = \{(x, y)\}$ forms a vector space with respect to the component-wise of addition and multiplication by scalars i.e $(x_1, x_2) + (y_1, y_2) = (x_1+y_1, x_2+y_2)$

V is a vector space Check!

2. Let V be the set $\mathbb{N} = \{1, 2, 3, \dots\}$. Is V a vector space under the two operations of addition $\oplus$ and scalar multiplication $\odot$?

Sol.

Since there is no $-\bar{x} \in V$ and no $\bar{0} \in V$ $\forall \bar{x} \in V$ axioms A_4 and A_5 have not been satisfied. Therefore V is not a vector space.

3. Consider $\mathbb{R}^n$ where addition $\oplus$ and scalar multiplication $\odot$ are defined in the usual manner. Is $\mathbb{R}^n$ a vector space? i.e

$$(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1+y_1, x_2+y_2, \dots, x_n+y_n)$$

and

$$\lambda(x_1, x_2, \dots, x_n) = (\lambda x_1, \lambda x_2, \dots, \lambda x_n)$$

$$\forall \lambda \in \mathbb{R}, x, y \in V.$$

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(4) Let V be the set of all ordered triple of the form (x_1, x_2, x_3) where addition $\oplus$ and scalar multiplication $\odot$ are defined in the usual way by

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1+y_1, x_2+y_2, x_3+y_3) \text{ and}$$

$$c(x_1, x_2, x_3) = (cx_1, x_2, x_3)$$

Is V a vector space under the two operations?

Soln.

It is easily checked that axiom (D_1) is not satisfied since for any $\lambda, \mu \in \mathbb{R}$ and $\bar{x} \in V$

$$(\lambda+\mu)\bar{x} = (\lambda+\mu)(x_1, x_2, x_3)$$
$$= [(\lambda+\mu)x_1, x_2, x_3]$$

But $\lambda\bar{x} + \mu\bar{x} = \lambda(x_1, x_2, x_3) + \mu(x_1, x_2, x_3)$

$$= (\lambda x_1, x_2, x_3) + (\mu x_1, x_2, x_3)$$
$$= ((\lambda+\mu)x_1, 2x_2, 2x_3)$$

$$\therefore (\lambda+\mu)\bar{x} \neq \lambda\bar{x} + \mu\bar{x}.$$

Hence V is not a vector space under the two operators of $\oplus$ and $\odot$.

Exercises

(1) Show that $V = \mathbb{R}^2$ is not a vector space over $\mathbb{R}$

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with respect to the operations of vector addition 5
and scalar multiplication defined as .

$$(x,y)+(z,t)=(x+z,\ y+t) \text{ and}$$

$$k(x,y)=(k^2x,\ ky)\ldots$$

② Show that the set of $M_{2\times 2}(\mathbb{R})=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}, a,b,c,d\in\mathbb{R}\right\}$ forms a vector space with the usual vector addition and scalar multiplication.

③ Show that the set of $C[a,b]=\{f:[a,b]\longrightarrow\mathbb{R} : f \text{ is continuous}\}$ of all continuous functions defined on $[a,b]$ do forms a vector space. Similarly $C^m[a,b]=\{f:[a,b]\longrightarrow\mathbb{R}\}$ do forms a vectorspace ie

$$(f+g)'(x)=f'(x)+g'(x),$$

$$(\lambda f')(x)=\lambda f'(x).$$

$$\boxed{\begin{aligned}(f+g)(x)&=f(x)+g(x)\\ \lambda(f+g)(x)&=(\lambda f+\lambda g)(x)\end{aligned}}$$

④ Prove with standard operations in $\mathbb{R}^2$, the set $V=\{(x,3x): x\in\mathbb{R}\}$ is a vector space.

⑤ Let V be the set of all ordered pairs of real numbers with addition defined by $(x_1,x_2)+(y_1,y_2)=(x_1+y_1,\ x_2+y_2)$ and scalar multiplication defined by $\alpha\cdot(x_1,x_2)=(\alpha x_1,\ x_2)$
Is V a vector space? Justify your answer.

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Subspace of a Vector Space

Defn :- Let V be a vector space and W is a subset of V, if W is a vector space w.r.t the operations of addition and multiplication in V, then W is called a subspace of V.

Obviously V is a subspace of itself and the single element $\{0\}$ is also a subspace of V because it satisfied all the axioms of a vector space. Thus, every vector space has at least two subspaces and these we called the trivial subspaces.

Note that it is not necessary to check all 10 axioms of the defining condition in order to determine if a subset is also a subspace only the closure condition (A_1) and (M_1) need to be considered. That is, a non-empty subset W of vector space V is a subspace of V iff.

$$A_1 :- \bar{x}+\bar{y} \in W \text{ and } \forall \bar{x},\bar{y} \in W$$

$$M_1 :- \lambda\bar{x} \in W, \ \lambda \in \mathbb{R}, \ \bar{x} \in V.$$

Theorem :- W is a subspace of V if and only if

(i) W is nonempty (or; $0 \in W$).

(ii) W is closed under vector addition, i.e

$\forall \bar{x},\bar{y} \in W \Rightarrow \bar{x}+\bar{y} \in W$

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(iii) W is closed under scalar multiplication $\bar{x} \in W$

ie $\forall \bar{x} \in W \Rightarrow \lambda\bar{x} \in W$ for any $\lambda \in \mathbb{R}$.

Proof :-

If W is a subset of V, then W is automatically ~~is~~ inherits all the vectors space properties of V except (A_1), (A_4), (A_5) and (M_1).

However, (A_1) togather with (M_1) implies (A_4) & (A_5).

To prove this, Observe that

$$(M_1) \Rightarrow -\bar{x} = (-1)\bar{x} \in W \quad \forall \bar{x} \in W, \forall -1 \in \mathbb{R}$$

so that (A_5) holds. Since $(-\bar{x}) \in W$, (A_1) ensures that $\bar{x} + (-\bar{x}) = 0 \in W$.

Example (1) : Let $V = \mathbb{R}^3$ and let W be a subset of all vectors in $\mathbb{R}^3$ of the form $(x_1, x_2, 0)$. Is W a subspace of V?

Soln.

We shall first see whether W is closed under the two operations of addition (+) and scalar multiplication (·).

Let $\bar{x} = (x_1, x_2, 0)$ and $\bar{y} = (y_1, y_2, 0)$, $\forall \bar{x}, \bar{y} \in W$.

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Now, $\bar{x}+\bar{y} = (x_1, x_2, 0) + (y_1, y_2, 0)$

$= (x_1+y_1, x_2+y_2, 0) \in W$ (closure under addition)

Also, $\lambda.\bar{x} = \lambda(x_1, x_2, 0)$

$= (\lambda x_1, \lambda x_2, 0) \in W$ (closure under scalar multiplecation).

We can similarly the rest of the axioms and see that W is a vector space. Hence W is a subspace of V.

Example (2) :- Let $V = \mathbb{R}^3$ and let W be of the form $(x_1, x_2, 1)$. Is W a subspace of V?

Soln.

First we check whether W is closed under the two operations ⊕ and ⊙ before going over the rest of the axiom of vector space.

Let $\bar{x} = (x_1, x_2, 1)$ and $\bar{y} = (y_1, y_2, 1)$

Now, $\bar{x}+\bar{y} = (x_1, x_2, 1) + (y_1, y_2, 1)$

$= (x_1+y_1, x_2+y_2, 2) \notin W$

We observe that the third compoment is 2 and not 1 which shows that W is not closed under addition. Therefore W is not a subspace of V.

Example (3)

(a) Show that W is not a subspace of $V = \mathbb{R}^3$ where W consist of those elements whose length does not

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exceed 1. i.e. $W = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$

$\bar{x} = (1, 0, 0)$, $\bar{y} = (0, 1, 0)$,

Hence W is not a subspace of V. since

$$\bar{x} + \bar{y} = (1, 0, 0) + (0, 1, 0) = (1, 1, 0) \notin W.$$

Theorem :-

Let V be vector space with operations of addition $\oplus$ and scalar multiplication $\odot$ and let W be a non-empty subset of V. Then W is a subspace of V iff

$$\lambda \bar{x} + \bar{y} \in W \quad \forall \bar{x}, \bar{y} \in V, \lambda \in \mathbb{R} \quad \text{———} \ (*)$$

Proof :- $(\Rightarrow)$

Suppose W is a subspace, then $(*)$ hold since W is a subspace and must satisfies all the axioms of vector space in V. For if $\bar{x}, \bar{y} \in W$, $\lambda \in \mathbb{R}$ then,

$$\lambda \bar{x} + \bar{y} \in W \quad \forall \bar{x}, \bar{y} \in V, \lambda \in \mathbb{R}$$

(closure under addition

and

$\lambda \bar{x} \in W \quad \forall \bar{x} \in V, \lambda \in \mathbb{R}$ (closure under scalar multiplication).

$(\Leftarrow)$ conversely, suppose that $(*)$ holds, then we prove that W is a subspace of V by showing that $(*)$ satisfie

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the axioms of vector space.

Now take $\lambda = 1$, then $\lambda\bar{x} + \bar{y} = \bar{x} + \bar{y} \in W, \forall \bar{x}, \bar{y} \in$

(closure under addition)

axioms of associativity and commutativity laws under addition holds. Since $\bar{x}, \bar{y} \in W$.

Take $\lambda = -1$ and let $\bar{x} = \bar{y}$, then

$$\lambda\bar{x} + \bar{y} = -\bar{x} + x = 0 \in W \quad \forall \bar{x}, \bar{y} \in W$$

(existence of identity)

Next, take $\bar{y} = \bar{0}$, then,

$$\lambda\bar{x} + \bar{y} = \lambda\bar{x} + \bar{0} = \lambda\bar{x} \in W$$ (closure under scalar multiplication)

Axioms $(M_2), (M_3), (\Delta_1)$ and (Δ_2) holds because vectors in W are vectors in V. Therefore W is a vector space. This shows that W is a subspace of the vector space V.

<u>Examples</u>. Use the theorem above to verify the results in example ① and ②.

<u>Soln</u>.

$$\lambda\bar{x} + \bar{y} = \lambda(x_1, x_2, 0) + (y_1, y_2, 0)$$
$$= (\lambda\bar{x}_1 + y_1, \lambda x_2 + y_2, 0) \in W \text{ is true.}$$

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Also,

$$\lambda\bar{x} + \bar{y} = \lambda(x_1, x_2, 1) + (y_1, y_2, 1)$$
$$= (\lambda x_1 + y_1, \lambda x_2 + y_2, \lambda + 1) \notin W$$

is true.

<u>Example</u> :- Let W be a subset of $\mathbb{R}^4$ consisting of all vectors of the form $(a, b, a-b, a+2b)$ where $a, b \in \mathbb{R}$. Show that W is a subspace in $\mathbb{R}^4$.

<u>Solution</u>.

Let $\bar{x} = (x_1, x_2, x_1 - x_2, x + 2x_2)$ and

$\bar{y} = (y_1, y_2, y_1 - y_2, y_1 + 2y_2)$

We want to show that

$\lambda\bar{x} + \bar{y} \in W \ \forall \bar{x}, \bar{y} \in V, \lambda \in \mathbb{R}$.

$$\lambda\bar{x} + \bar{y} = \lambda(x_1, x_2, x_1 - x_2, x_1 + 2x_2) + (y_1, y_2, y_1 - y_2, y_1 + 2y_2)$$
$$= (\lambda x_1, \lambda x_2, \lambda(x_1 - x_2), \lambda(x_1 + 2x_2)) + (y_1, y_2, y_1 - y_2, y_1 + 2y_2)$$
$$= \lambda x_1 + y_1, \lambda x_2 + y_2, \lambda(x_1 - x_2) + (y_1 - y_2), \lambda(x_1 + 2x_2) + (y_1 + 2y_2)$$
$$= \lambda x_1 + y_1, \lambda x_2 + y_2, (\lambda x_1 + y_1) - (\lambda x_2 + y_2), (\lambda x_1 + y_1) + 2(\lambda x_2 + y_2)$$
$$= (a, b, a - b, a + 2b) \in W$$

Where $a = \lambda x_1 + y_1$ and $b = \lambda x_2 + y_2$.

Hence W is a subspace in $\mathbb{R}^4$.

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exercises.

① Let $V = \mathbb{R}^3$ and Let $W = \{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\}$ show that W is a subspace of V.

② Let $V = \mathbb{R}^2$ and $W := \{(2x_1, 3x_2) : x_1, x_2 \in \mathbb{R}\}$ show that W is a subspace of V.

Linear Combination.

Defn:- Let $S = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ be a set of vectors in a vector space V. Then a vector $\bar{x} \in V$ is said to be a linear combination of the vectors in S if $\bar{x} = c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n$ form some scalars $c_1, c_2, \ldots, c_n$.

Example ①: Let $V = \mathbb{R}^4$ and let $S = \{\bar{x}_1, \bar{x}_2, \bar{x}_3\} \in V$ where $\bar{x}_1 = (1,2,1,-1)$, $\bar{x}_2 = (1,0,2,-3)$ and $\bar{x}_3 = (1,1,0,-2)$, show that $\bar{x} = (2,1,5,-5)$ is a linear combination of the vectors in S..

Solution.

We can find some scalars c_1, c_2, and c_3 such that
$\bar{x} = c_1\bar{x}_1 + c_2\bar{x}_2 + c_3\bar{x}_3$

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By substituting $\bar{x}_1, \bar{x}_2, \bar{x}_3$ and $\bar{x}$ into the equation we have.

$$(2,1,5,-5) = c_1(1,2,1,-1) + c_2(1,0,2,-3) + c_3(1,1,0,-2)$$

Equating the corresponding coefficients will lead to the linear system below.

$$c_1 + c_2 + c_3 = 2 \quad \text{—— (1)}$$

$$2c_1 + c_3 = 1 \quad \text{—— (2)}$$

$$c_1 + 2c_2 = 5 \quad \text{—— (3)}$$

$$-c_1 - 3c_2 - 2c_3 = -5 \quad \text{—— (4)}$$

Add (1) and (4) we have

$$-2c_2 - c_3 = -3$$

$$\therefore c_3 = 3 - 2c_2 \quad \text{—— (5)}$$

substitute (5) in (2) we have.

$$2c_1 + 3 - 2c_2 = 1$$

$$2c_1 - 2c_2 = -2$$

$$2(c_1 - c_2) = -2$$

$$c_1 - c_2 = -1$$

$$\therefore c_1 = c_2 - 1 \quad \text{—— (6)}$$

substitute (6) into (3) we have

$$c_2 - 1 + 2c_2 = 5$$

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$3C_2 = 6$

$\therefore C_2 = 2$

$C_3 = 3 - 2(2)$
$= -1$

$C_1 = 2 - 1$
$= 1$

Having to solve system and obtained $C_1 = 1$, $C_2 = 2$, and $C_3 = -1$. This shows that $\bar{x}$ is a linear combination of the vectors $\bar{x}_1, \bar{x}_2, \bar{x}_3$ e.g.

Example 2

Write the vectors $U = (1, -2, 5)$ as a linear combination of the vectors $e_1 = (1, 1, 1)$, $e_2 = (1, 2, 3)$ and $e_3 = (2, -1, 1)$

Soln

We wish to express U as $U = xe_1 + ye_2 + ze_3$ with $x, y,$ and z no yet unknown scalars. Thus we require

$$(1, -2, 5) = x(1, 1, 1) + y(1, 2, 3) + z(2, -1, 1)$$

$$= (x, x, x) + (y, 2y, 3y) + (2z, -z, z)$$

$$= (x + y + 2z, x + 2y - z, x + 3y + z)$$

From the equivalent system of equations by setting the corresponding components equal to each other, and then reduce to echelon form.

$x + y + 2z = 1$
$x + 2y - z = -2$
$x + 3y + z = 5$

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Subtract row 1 from row 2 and row 3 we have (15)

$$x + y + 2z = 1$$
$$y - 3z = -3$$
$$2y - z = 4$$

$\boxed{\begin{array}{l} R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1 \end{array}}$

-2 Row 2 and add with Row 3 we have

$$x + y + 2z = 1$$
$$y - 3z = -3$$
$$5z = 10$$

$\boxed{R_3 = R_3 + (-2R_2)}$

Note that the above system is consistent and so has a solution. Solve for the unknowns to obtain $x = -6$, $y = 3$, $z = 2$. Hence

$$U = -6e_1 + 3e_2 + 2e_3$$

Exercise

Express $p = 3t^2 + 5t - 5$ as linear combination of P_1, P_2 \$ P_3 where; $P_1 = t^2 + 2t + 1$, $P_2 = 2t^2 + 5t + 4$ and $P_3 = t^2 + 3t + 6$

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Span (or generate) of a Vectors Space V.

16

Defn:- A Set $S = \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\}$ of vectors in a vector space V is said to Span V if every vector in V is a linear combination of vectors in S. i.e. every $y_i \in V$ is a linear combination of vectors in S.

Example (1)

Let $V = \mathbb{R}^3$ and Let $S = \{\bar{x}_1, \bar{x}_2, \bar{x}_3\} \in V$ where $\bar{x}_1 = (1, 1, 2)$, $\bar{x}_2 = (1,0,2)$, $\bar{x}_3 = (1,1,0)$. Does S span V

Solution.

Take any arbitrary vector $\bar{x} = (t_1, t_2, t_3) \in V$ and then find out whether there are some constants c_1, c_2, c_3 such that

$$\bar{x} = c_1\bar{x}_1 + c_2\bar{x}_2 + c_3\bar{x}_3$$

Thus,
$$(t_1, t_2, t_3) = c_1(1,1,2) + c_2(1,0,2) + c_3(1,1,0)$$

equate the corresponding coefficients, we have

$$(t_1, t_2, t_3) = (c_1, c_1, 2c_1) + (c_2, 0, 2c_2) + (c_3, c_3, 0)$$
$$= (c_1 + c_2 + c_3,\ c_1 + c_3,\ 2c_1 + 2c_2)$$

This implies

$$c_1 + c_2 + c_3 = t_1 \quad \text{--- (1)}$$
$$c_1 + c_3 = t_2 \quad \text{--- (2)}$$
$$2c_1 + 2c_2 = t_3 \quad \text{--- (3)}$$

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From ①

$$C_3 = t_1 - C_1 - C_2 \quad \text{——} \quad ④$$

substitute ④ in ②

$$C_1 + t_1 - C_1 - C_2 = t_2$$

$$t_1 - C_2 = t_2$$

$$\therefore C_2 = t_1 - t_2 \quad \text{——} \quad ⑤$$

substitute ⑤ in ③

$$2C_1 + 2(t_1 - t_2) = t_3$$

$$2C_1 + 2t_1 - 2t_2 = t_3$$

$$2C_1 = t_3 + 2t_2 - 2t_1$$

$$\therefore C_1 = \tfrac{1}{2}t_3 + t_2 - t_1 \quad \text{——} \quad ⑥$$

finally substitute ⑤ and ⑥ in ④

$$C_3 = t_1 - (\tfrac{1}{2}t_3 + t_2 - t_1) - (t_1 - t_2)$$

$$= t_1 - \tfrac{1}{2}t_3$$

By solving the system in terms of t as $C_1 = \frac{1}{2}t_3 + t_2 - t_1$, $C_2 = t_1 - t_2$ and $C_3 = t_1 - \frac{1}{2}t_3$. That is, having found the scalars C_1, C_2 and C_3, we therefore conclude that S spans $V = \mathbb{R}^3$.

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Example ② :- Show that the vectors $U=(1,2,3)$, $V=(0,1,2)$ and $W=(0,0,1)$ span $\mathbb{R}^3$

Solution.

We need to show that an arbitrary vector $(a,b,c)\in\mathbb{R}^3$ is a linear combination of U, V, and W.

Set $(a,b,c) = xU + yV + zW$

$$= x(1,2,3) + y(0,1,2) + z(0,0,1)$$

$$= (x,2x,3x) + (0,y,2y) + (0,0,z)$$

$$= (x,\ 2x+y,\ 3x+2y+z)$$

By equating the corresponding we have

$$\begin{aligned} x &= a \\ 2x+y &= b \\ 3x+2y+z &= C \end{aligned} \quad \text{or} \quad \begin{aligned} 3x+2y+z &= c \\ 2x+y &= b \\ x &= a. \end{aligned}$$

This implies $x=a$, $y=b-2a$ and

$$z = c-2(b-2a)-3a$$
$$= c-2b+a$$

is a solution. Hence U, V and W span $\mathbb{R}^3$

Exercise.

Show that $U_1=(1,2,5)$, $U_2=(1,3,7)$ and $U_3=(1,-1,-1)$ do not span $\mathbb{R}^3$.

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Linear Dependence

Defn:- Let V be a vector space over a field K. The vectors $v_1, v_2, \ldots, v_m \in V$ are said to be linearly dependent over K, or simply dependent, if there exists scalars $a_1, a_2, \ldots, a_m \in K$ not all of them 0, such that

$$a_1v_1 + a_2v_2 + \cdots + a_mv_m = 0 \quad \text{———} \quad (*)$$

Otherwise, the vectors are said to be linearly independent over K, or simply independent.

Remark:-

Observe that the relation in $(*)$ will always hold if the a's are all 0. If this relation holds when one of the a's is not 0, then the vectors are linearly dependent.

Defn:- A set $\{v_1, v_2, \ldots, v_m\}$ is called a dependent or independent set according as the vectors $\{v_1, v_2, \cdots v_m\}$ are linearly dependent or independent.

An infinite set S of vectors is linearly dependent if there exists vectors $u_1, u_2, \ldots, u_k$ in S which are linearly dependent; otherwise S is linearly independent.

The empty set ϕ is defined to be linearly independent.

Note:- Two vectors v_1 and v_2 are dependent if and only if one of them is a multiple of the other.

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Examples.

① Determine whether U and V are linearly dependent where

ⓐ $U = (3, 4)$, $V = (1, -3)$, ⓑ $U = (2, -3)$ $V = (6, -9)$.

Soln.

ⓐ No,

ⓑ Yes, $U = 2V$.

② Determine whether U and V are linearly dependent where ⓐ $U = (4, 3, -2)$ $V = (2, -6, 7)$

ⓑ $U = (-4, 6, 2)$ $V = (2, -3, 1)$

Soln.

ⓐ No, neither is a multiple of the other.

ⓑ Yes, $U = -2V$.

③ Determine whether the matrices A and B are dependent where ⓐ $A = \begin{pmatrix} 1 & -2 & 4 \\ 3 & 0 & -1 \end{pmatrix}$ $B = \begin{pmatrix} 2 & -4 & 8 \\ 6 & 0 & -2 \end{pmatrix}$

ⓑ $A = \begin{pmatrix} 1 & 2 & -3 \\ 6 & -5 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 6 & -5 & 4 \\ 1 & 2 & -3 \end{pmatrix}$

④ Determine whether the polynomials U and V are dependent where

ⓐ $U = 2 - 5t + 6t^2 - t^3$ / $V = 3t + 2t - 4t^2 + 5t^3$

ⓑ $U = 1 - 3t + 2t^2 - 3t^3$ / $V = -3 + 9t - 6t^2 + 9t^3$

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(5) Determine whether or not the vectors $(1,-2,1)$, $(2,1,-1)$, $(7,-4,1)$ are linearly dependent.

Solution.

Set a linear combination of the vectors equal to the zero vector using unknown scalars x, y and z

$$x(1,-2,1)+y(2,1,-1)+z(7,-4,1)=(0,0,0)$$
$$(x,-2x,x)+(2y,y,-y)+(7z,-4z,z)=(0,0,0)$$

Set the corresponding components equal to each other to obtain the equivalent homogeneous system and reduce to echelon form.

$$\begin{aligned} x+2y+7z&=0 \\ -2x+y-4z&=0 \\ x-y+z&=0 \end{aligned} \qquad \begin{aligned} R_2&=R_2+2R_1 \\ R_3&=R_3-R_1 \end{aligned}$$

$$\begin{aligned} x+2y+7z&=0 \\ 5y+10z&=0 \Rightarrow y+2z=0 \\ -3y-6z&=0 \Rightarrow y-2z=0 \end{aligned}$$

$$\begin{aligned} x+2y+7z&=0 \\ y+2z&=0 \end{aligned}$$

The system in echelon form, has only two nonzero equations in three unknowns; hence the system has a nonzero ~~Solut~~ solution. Thus the original vectors are linearly dependent.

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(6) Determine whether $(1,2,-3)$, $(1,-3,2)$, $(2,-1,5)$ are linearly dependent.

Soln.

$$x(1,2,-3)+y(1,-3,2)+z(2,-1,5)=(0,0,0)$$

This gives

$$\begin{aligned} x+y+2z&=0 \\ 2x-3y-z&=0 \\ -3x+2y+5z&=0 \end{aligned} \quad \Bigg| \quad \begin{aligned} x+y+2z&=0 \\ -5y-5z&=0 \\ 5y+11z&=0 \end{aligned}$$

$$\boxed{\begin{aligned} R_2&=R_2-2R_1 \\ R_3&=R_3+3R_1 \end{aligned}} \qquad \boxed{R_3=R_3+R_2}$$

$$\Rightarrow \quad \begin{aligned} x+y+2z&=0 \\ y+z&=0 \\ 6z&=0 \end{aligned}$$

Since the homogeneous has three nonzero equation in 3 unknown, the vectors are independent.

Exercises

(1) Determine whether the matrices A, B and C are dependent, where $A=\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$, $B=\begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}$, $C=\begin{pmatrix} 1 & -5 \\ -4 & 0 \end{pmatrix}$

(2) Let V be the vector space of polynomials of degree 3 over $\mathbb{R}$. Determine whether $u=t^3-3t^2+5t+1$

$v=t^3-t^2+8t+2$

$w=2t^3+4t^2+9t+5$

are linearly dependent or independent.

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BASIS

Defn:- A Set $S = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ of vectors, in a vector space V is called a basis for V if S span V and S is linearly independent.

example ①

Determine whether or not $S = \{(1,0),(0,1)\}$ is a basis in $\mathbb{R}^2$

Soln.

Let (x_1, x_2) be a vector in $\mathbb{R}^2$. We are to find scalars C_1 and C_2 such that

$$C_1(1,0) + C_2(0,1) = (x_1, x_2)$$

$$(C_1, 0) + (0, C_2) = (x_1, x_2)$$

$$C_1 + 0 = x_1 \Rightarrow C_1 = x_1$$

$$0 + C_2 = x_2 \Rightarrow C_2 = x_2$$

Since we have found the scalars C_1 and C_2 it means every vector in $\mathbb{R}^2$ is a linear combination of vectors in S. Therefore S span $\mathbb{R}^2$.

Next we show that S is linearly independent.

$$\text{Let } C_1(1,0) + C_2(0,1) = (0,0)$$

$$C_1 + 0 = 0 \Rightarrow C_1 = 0$$

$$0 + C_2 = 0 \Rightarrow C_2 = 0$$

Hence S is linearly independent. Since the Set S spans $\mathbb{R}^2$ and is linearly independent. So S is a basis in $\mathbb{R}^2$

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Example (2):

Find a basis for $\mathbb{R}^4$ which contains the vectors $(1,0,1,0)$ and $(-1,1,-1,0)$

Solution:

Goal:- We are to show that the two vectors span $\mathbb{R}^4$ and are linearly independent.

Let $\bar{x} \in \mathbb{R}^4$ such that $\bar{x} = (t_1, t_2, t_3, t_4)$. Then

$$C_1(1,0,1,0) + C_2(-1,1,-1,0) = (t_1, t_2, t_3, t_4)$$

$$(C_1, 0, C_1, 0) + (-C_2, C_2, -C_2, 0) = (t_1, t_2, t_3, t_4)$$

Now, equate the corresponding coefficients, we have

$$C_1 - C_2 = t_1$$
$$C_2 = t_2$$
$$C_1 - C_2 = t_3$$

$\therefore C_2 = t_2$ and $C_1 = t_1 + t_2$.

We have found scalars C_1 and C_2 such that the vector $\bar{x}$ is a linear combination of vectors in $S = \{(1,0,1,0), (-1,1,-1,0)\} \in \mathbb{R}^4$.

Again, Let $C_1(1,0,1,0) + C_2(-1,1,-1,0) = (0,0,0,0)$

$$(C_1, 0, C_1, 0) + (-C_2, C_2, -C_2, 0) = (0,0,0,0)$$

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$c_1 - c_2 = 0$
$c_2 = 0$ $\Rightarrow c_1 = c_2 = 0$

S is linearly independent. Hence S is a basis for $\mathbb{R}^4$.

exercise

① Determine whether or not $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis in $\mathbb{R}^3$.

② Find a basis for $\mathbb{R}^4$ which contains the vectors $\bar{x} = (-2,-1,1,0)$ and $\bar{y} = (1,-1,-3,5)$.

③

DIMENSION

Defn:- A vector space V is said to be of finite dimension n or to be n-dimensional, written as $\dim V = n$, if V contains a basis with n elements. The vector space $\{0\}$ is defined to have dimension 0. If V does not have a finite basis, it is said to be of infinite dimension.

Example ①

Find a basis and dimension of the subset W of $\mathbb{R}^4$ spanned by $(1,-4,-2,1)$, $(1,-3,-1,2)$ and $(3,-8,-2,7)$.

solution

Reduce to echelon form the matrix whose rows are

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the given vectors

$$\begin{pmatrix} 1 & -4 & -2 & 1 \\ 1 & -3 & -1 & 2 \\ 3 & -8 & -2 & 7 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & -4 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 4 & 4 & 4 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & -4 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The nonzero rows in the echelon matrix form $(1,-4,-2,1)$ and $(0,1,1,1)$ form a basis of W and so $\dim W = 2$

Exercise

Find a basis and the dimension of the subspace W of $\mathbb{R}^4$ spanned by $(1,4,-1,1)$, $(0,1,-3,-1)$ and $(0,2,1,-5)$.

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Linear Transformation

Defn:- A transformation $T: V \rightarrow W$ is a rule that assign each vector in V exactly one vector in W where V is the domain, W is codomain.

For all $v \in V$ $T(V)$ is called image of V in W.

The set of all image of U, $(T(U))$ is called range.

Defn:- T as define above is linear if

(i) $T(u+v) = T(u) + T(v)$, $\forall\, u, v \in V$

(ii) $T(ku) = kT(u)$, where $k \in \mathbb{R}$, $u \in V$.

Example:- Determine whether the following transform are linear

(a) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x+y \\ 2x \end{pmatrix}$

(b) $T(x, y) = (x+y, y+z)$

(a) **Solution** Given that

$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x+y \\ 2x \end{pmatrix}$, $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

Let $u, v \in \mathbb{R}^2$ such that $u = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $V = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ and

k be a scalar.

Goal:- (i) $T(u+v) = T(u) + T(v)$ (ii) $T(ku) = kT(u)$.

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(i) $T(u+v) = T\left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right]$

$= T\left[\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}\right]$

$= \begin{pmatrix} (x_1+x_2) - (y_1+y_2) \\ (x_1+x_2) + (y_1+y_2) \\ 2(x_1+x_2) \end{pmatrix}$

$= \begin{pmatrix} (x_1-y_1) + (x_2-y_2) \\ (x_1+y_1) + (x_2+y_2) \\ 2x_1 + 2x_2 \end{pmatrix}$

$= \begin{pmatrix} x_1 - y_1 \\ x_1 + y_1 \\ 2x_1 \end{pmatrix} + \begin{pmatrix} x_2 - y_2 \\ x_2 + y_2 \\ 2x_2 \end{pmatrix}$

$= T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

$= T(u) + T(v).$

(ii) $T(ku)$

$= T\left(k\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right)$

$= T\begin{pmatrix} kx_1 \\ ky_1 \end{pmatrix}$

$= \begin{pmatrix} kx_1 - ky_1 \\ kx_1 + ky_1 \\ 2kx_1 \end{pmatrix}$

$= k\begin{pmatrix} x_1 - y_1 \\ x_1 + y_1 \\ 2x_1 \end{pmatrix}$

$= kT\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

$= kT(u).$

Hence T is linear.

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(b) Solution

Given that $T(x,y) = (x+y, y+z)$, $T: \mathbb{R}^2 \to \mathbb{R}^2$

Let $u = (x_1, y_1)$, $v = (x_2, y_2)$. Goal (i) $T(u+v) = T(u) + T(v)$.
(ii) $T(ku) = kT(u)$.

(i) $T(u+v) = T(x_1, y_1 + (x_2 y_2)$

$= T(x_1 + x_2, y_1 + y_2)$

$= (x_1 + x_2 + y_1 + y_2, y_1 + y_2 + 2)$

Now,

$T(u) = T(x_1, y_1) = (x_1 + y_1, y_1 + 2)$

$T(v) = T(x_2, y_2) = (x_2 + y_2, y_2 + 2)$

$T(u) + T(v) = (x_1 + y_1 + x_2 + y_2, y_1 + y_2 + 4)$

$\therefore T(u+v) \neq T(u) + T(v)$.

(ii) $T(ku) = T(k(x_1, y)$

$= T(kx_1, ky_1)$

$= (kx_1 + ky_2, ky_1 + 2k)$

Now,
$kT(u) = k(x_1 + y_1, y_1 + 2)$

$= (kx_1 + ky_1, ky_1 + 2k)$. $\therefore T(ku) = kT(u)$

Hence T is not linear transform

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Defn:- Let $T: V \longrightarrow W$ be a linear Transformation and V is every vector in V expressed in form of

$$V = K_1V_1 + K_2V_2 + \cdots + K_mV_m \quad \text{then}$$

$$\begin{aligned} T(V) &= T(K_1V_1 + K_2V_2 + \cdots + K_mV_m) \\ &= T(K_1V_1) + T(K_2V_2) + \cdots + T(K_mV_m) \\ &= K_1T(V_1) + K_2T(V_2) + \cdots + K_mT(V_m). \\ &= T(V_1)\ T(V_2) \cdots T(V_m) \cdot \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_m \end{pmatrix} \end{aligned}$$

$$T(V) = AV$$

A matrix A is called linear transformation matrix from V to W whose columns are $T(V_1), T(V_2), \cdots T(V_m)$.
If $V_1, V_2, \ldots, V_m$ are standard basis elements of vector in V then A a standard matrix representation of T.

Example:- The linear transformation $T: \mathbb{R}^3 \to \mathbb{R}$ defined by $T(x_1, x_2, x_3)^T = (x_1+x_2, x_2+x_3)^T$ Show that T is linear and hence find a matrix A such that $T(x_1, x_2, x_3)^T = A(x_1, x_2, x_3)^T$.

Solution

Given $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ x_2+x_3 \end{pmatrix}$, Goal: (i) $T(u+v) = T(u) + T(v)$
(ii) $T(Ku) = KT(u)$.

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Let $u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3)$

i) $T(u+v) = T\left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right]$

$= T\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}$

$= \begin{pmatrix} x_1 + y_1 + x_2 + y_2 \\ x_2 + y_2 + x_3 + y_3 \end{pmatrix}$

Now.

$T(u) = T(x_1, x_2, x_3) = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}$

$T(v) = T(y_1, y_2, y_3) = \begin{pmatrix} y_1 + y_2 \\ y_2 + y_3 \end{pmatrix}$

$T(u) + T(v) = \begin{pmatrix} x_1 + x_2 + y_1 + y_2 \\ x_2 + x_3 + y_2 + y_3 \end{pmatrix}$ $\therefore T(u+v) = T(u) + T(v)$

(ii) $T(ku) = T\left(k\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right)$

$= T\begin{pmatrix} kx_1 \\ kx_2 \\ kx_3 \end{pmatrix}$

$= \begin{pmatrix} kx_1 + kx_2 \\ kx_2 + kx_3 \end{pmatrix}$

Now,

$kT(u) = k\begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}$

$= \begin{pmatrix} kx_1 + kx_2 \\ kx_2 + kx_3 \end{pmatrix}$

$\therefore T(ku) = kT(u)$

Hence T is a linear transform

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$A = \left(T(v_1), T(v_2), T(v_3) \right)$, where $v_1 = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$,

$v_2 = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $v_3 = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$$T(v_1) = T(e_1) = T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$T(v_2) = T(e_2) = T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$T(v_3) = T(e_3) = T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$\therefore A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Then, $T(x_1, x_2, x_3) = A(x_1, x_2, x_3)$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}.$$

~~Exercise~~ (7). (a) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be transformation defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$ Show that T is linear

(b) Find the standard matrix representation for the linear transformation defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - 3z \\ -x + 3y \\ 3y - z \end{pmatrix}$

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exercise (2) → The standard representation of

Find the standard matrix A such that

$$T(X) = (3x_1 - x_3, -2x_1 + 4x_2, 6x_2 - 3x_3)^T \quad \text{ie}$$

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 - x_3 \\ -2x_1 + 4x_2 \\ 6x_2 - 3x_3 \end{pmatrix}.$$

Exercise (7)

Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be transformation defined by.

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x + 2y \\ 6x \\ 5x - 2y \end{pmatrix}$$ Show that T is linear

and hence find the stand matrix representation for the linear transformation defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9x_1 - 3x_3 \\ -6x_1 + 12x_2 \\ 18x_2 - 9x_3 \end{pmatrix}.$$

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MATRICES

1) **MATRIX**: A Matrix is a set of numbers arranged in rows and columns to form a rectangular array. A Matrix having m rows and n columns is called an $(m \times n)$ Matrix.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Example $\begin{bmatrix} 3 & 5 & 6 \\ 4 & 2 & 1 \end{bmatrix}$ is a 2×3 matrix.

2) **Transpose of a Matrix**: If the rows and columns of a matrix are interchanged, the new matrix formed is called the transpose of the original Matrix.

Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \text{Then } A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

3) **Square Matrix**: is a Matrix that has equal number of rows and columns. e.g $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ 2×2.

4) **Determinant of a Square Matrix**: The determinant of a Square Matrix $A = [a_{ij}]$ is a number denoted by $\det(A)$ or $|A| = |a_{ij}|$.

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Example :- For a 2×2 Matrix A. (35)

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \times a_{22} - a_{21} \times a_{12}.$$

Similarly for a 3×3 matrix A we have.

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Give a particular Example pls.

$$= a_{11}[a_{22} \cdot a_{33} - a_{32} \cdot a_{23}] - a_{12}[a_{21} \cdot a_{23} - a_{31} \cdot a_{23}] + a_{13}[a_{21} \cdot a_{32} - a_{31} \cdot a_{22}]$$

The matrices $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ $\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ are called Sub-Matrices or Minors of the elements a_{11}, a_{12} and a_{13} respectively of the 3×3 Matrix.. ~~of a 4×4~~

~~To compute the determinant~~

These Minors have associated signs (+ or −) depending on the position in the determinant of the element of which they are minor.

To compute the determinant of a 4×4 Matrix first the determinant is reduced to 3×3 determinants of the elements a_{11}, a_{12}, a_{13} and a_{14} respectively. Then each of these determinants in turn is reduced to 2×2 minors maintaining all previous factors in each operation. this principle is extended to the determination of determinants of matrices of any dimension.

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(5) Cofactors: Each element in a Matrix give rise to a cofactor, which is the minor of the element in the determinant together with its "place sign". Generally the cofactor denoted by A_{ij} of the Matrix

$$A = (a_{ij})_{n \times n} \text{ is defined by } A_{ij} = (-1)^{i+j} |M_{ij}|$$

where $|M_{ij}|$ is the minor of (a_{ij}).

The minor is a Sub-matrix of order $(n-1) \times (n-1)$ obtained by deleting the i^{th} row and j^{th} column of A, multiplied by $(-1)^{i+j}$

Example

Let $A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}$, The cofactor are given as

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix} = 4.$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 4 & 6 \\ 7 & 2 \end{vmatrix} = - \begin{vmatrix} 4 & 6 \\ 7 & 2 \end{vmatrix} = 34.$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 4 & 5 \\ 7 & 1 \end{vmatrix} = + \begin{vmatrix} 4 & 5 \\ 7 & 1 \end{vmatrix} = -31$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} = - \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} = 4.$$

$\vdots$

A_{33}

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(37)

(6) Adjoint of a Matrix:- If A is a Squ[are] matrix, then the adjoint of A, denoted by (adj A) is defined as the transpose of the Matrix of cofactors of A.

For Instance, Suppose $C = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

is a Matrix of cofactors of the matri[x]

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then the adjoint of [A]

is given by $\text{adj}(A) = C^T = \begin{bmatrix} A_{11} & A_{21} & A_{3[1]} \\ A_{12} & A_{22} & A_{3[2]} \\ A_{13} & A_{23} & A_{[33]} \end{bmatrix}$

Give any example please

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(7) The Inverse of a Matrix:- the inverse

of a square matrix A denoted by A^{-1} is defined as the adjoint of A divided by the determinant of A, written as $A^{-1} = \frac{Adj\,A}{|A|}$.

Example

(1) For order $n = 3$, let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then, $A^{-1} = \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

where $A_{ij} = (-1)^{i+j} |M_{ij}|$, $(i, j) = 1, 2 \cdots n$ are the cofactors of A.

(2) If $A = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, then $A^{-1} = -\frac{1}{8} \begin{bmatrix} 1 & -4 & 2 \\ -2 & 0 & 4 \\ 2 & 4 & -10 \end{bmatrix}$

Steps in Determining the Inverse of a Matrix

(i) Evaluate the determinant $|A|$

(ii) Write the ~~transpose of the~~ matrix of cofactors

(iii) write the transpose of the matrix of cofactors

(iv) Write the inverse as $A^{-1} = \frac{1}{|A|} \cdot C^{T}$, C^{T} is the transpose of the matrix of cofactors

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Some properties of Determinants (39)

① Rows and Column can be interchanged without affecting the value of the determinant:

$$|A| = |A^T|$$

② If a row (or column) is changed by adding to (or subtracting) its elements, the corresponding elements of any row (or column) of the determinant remains unchanged. e.g

$$\begin{vmatrix} 4 & 5 \\ 2 & -3 \end{vmatrix} = \begin{vmatrix} 4+2 & 5-3 \\ 2 & -3 \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 2 & -3 \end{vmatrix} = -2$$

i.e The first row is changed by adding the corresponding column elements of the second row.

③ If two rows or column are interchanged, the sign of the determinant is changed

e.g $\begin{vmatrix} 4 & 5 \\ 2 & -3 \end{vmatrix} = - \begin{vmatrix} 2 & -3 \\ 4 & 5 \end{vmatrix}$

④ If the element in any row (or column) ~~of a~~ of the determinant have a common factor λ, then the corresponding matrix ~~Matrix~~ equal the determinant of the which $\lambda = 1$

e.g $\begin{vmatrix} 8 & 10 \\ 2 & -3 \end{vmatrix} = 2 \begin{vmatrix} 4 & 5 \\ 2 & -3 \end{vmatrix} = 2 \times (-22) = -44$

⑤ When at least one row (or column) of a matrix is a linear combination of the other row (or column) the determinant is zero

e.g $\begin{vmatrix} 3 & 2 & 1 \\ 1 & 2 & -1 \\ 2 & - & 3 \end{vmatrix} = 0$

The determinant is zero because the first colmn is a linear combination of the second and the third column i.e column 1 = column 2 + column

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(6) The determinant of ~~the~~ an upper triangular or lower triangular matrix, is the product of main diagonal enteries.

e.g $\begin{vmatrix} 4 & 2 & 3 \\ 0 & 2 & -3 \\ 0 & 0 & 5 \end{vmatrix} = 4 \times 2 \times 5 = 40.$

(7) The determinant of the product of two square matrices is the product of the individual determinants

e.g $|AB| = |A||B|$ and $|A^2| = |A||A| = |A|^2$

$\Rightarrow |A^n| = |A|^n$ for any integer n.

(8) The determinant of a transpose of a matrix is the same as the original matrix

e.g $|A^T| = |A|$.

Matrix Methods for Solving System of equations

consider the set of linear equations below

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n &= b_m \end{aligned}\right\} \text{———} (1).$$

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this could be written in matrix form as.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad \text{——— (2)}$$

Def:- Matrix representation of a linear System:-
if A is the coefficient matrix of the System of linear equations (1) and b is a vector of constants, called column vector, then the linear System is expressed as.

$$Ax = b \quad \text{——— (3)}$$

Now $A^{-1}Ax = A^{-1}b$.

$$Ix = A^{-1}b$$

$$\Rightarrow x = A^{-1}b \quad \text{——— (4)}$$

Equation (4) is the Solution of the System(1)

Def: Singular and non-Singular Matrix

A Square matrix A is said to be non-Singular (or invertible) if the determinant of A

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is nonzero, and the rank of a non-singular $n \times n$ matrix is equal to n, written as rank(A) = n.
If however the determinant of A is zero, then the matrix is said to be <u>singular</u>.
This means that atleast one row and one column are dependent on the others. If the dependent row and column are removed, then we are left with an $(n-1)\times(n-1)$ matrix. Again if the determinant of this $(n-1)\times(n-1)$ matrix is still zero, we remove the dependent row and column to be left with an $(n-2)\times(n-2)$ matrix. Suppose that we eventually arrive at an $(r\times r)$ matrix whose determinant is not zero. Then the matrix A is said to ~~be~~ have <u>rank (r)</u> and we write rank (A) = r.

<u>Example</u>

(1) Since $|A| = \begin{vmatrix} 3 & 2 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 3 \end{vmatrix} = -3 \neq 0.$

$\Rightarrow \text{rank}(A) = 3.$

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(2) $|A| = \begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 0.$

Since the first row and column may be expressed as a linear combination of the others, we remove the first row and the first column to left with the determinant.

$$\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} = 5 \neq 0.$$

Then rank $(A) = \underline{2}$.

The inverse method of Solution

Example

(1) Solve the system by using the inverse method

$$2x + y + z = 5$$
$$x + 3y + 2z = 1$$
$$3x - 2y - 4z = -4$$

Soln

Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & -2 & -4 \end{bmatrix}$ be the matrix of the system and $b = \begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix}$ the coefficient of the column vector.

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Then the determinant of A is $|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & -2 & -4 \end{vmatrix} = -17$

Next we find the cofactors of the matrix A.

$A_{11} = \begin{vmatrix} 3 & 2 \\ -2 & -4 \end{vmatrix} = -8 \qquad A_{12} = -\begin{vmatrix} 1 & 2 \\ 3 & -4 \end{vmatrix} = 10 \qquad A_{13} = -11$

$A_{21} = 2 \qquad A_{22} = -11 \qquad A_{23} = 7 \qquad A_{31} = -1 \qquad A_{32} = -3$

$A_{33} = 5$

$$\therefore \operatorname{Adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} -8 & 10 & -11 \\ 2 & -11 & 7 \\ -1 & -3 & 5 \end{bmatrix}^T$$

$$= \begin{bmatrix} -8 & 2 & -1 \\ 10 & -11 & -3 \\ -11 & 7 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{\operatorname{Adj}(A)}{|A|} = -\frac{1}{17}\begin{bmatrix} -8 & 2 & -1 \\ 10 & -11 & -3 \\ -11 & 7 & 5 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 8 & -2 & 1 \\ -10 & 11 & 3 \\ +11 & -7 & -5 \end{bmatrix}$$

Going by the relation $x = A^{-1}b$.

we have.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{17}\begin{bmatrix} 8 & -2 & 1 \\ -10 & 11 & 3 \\ 11 & -7 & -5 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 34 \\ -51 \\ 68 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$$

$\therefore x = 2, \quad y = -3, \quad z = 4.$

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Cramer's Rule

Cramer's rule uses determinant, instead of inverse to solve linear systems of equations. Its major disadvantage is that you can only solve for one variable at a time. This is why most computer programs do not use this rule to solve systems of equations.

Let the simultaneous equations be as usual denoted as $Ax = b$; where A is a given $(n \times n)$ matrix, b is a given $(n \times 1)$ vector, and x is the $(n \times 1)$ vector of n unknowns. The explicit solution for the components $x_1, x_2 \dots x_n$ of x in terms of determinants is given as

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ b_2 & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & & & \\ b_n & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}}{|A|}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} & \cdots & a_{1n} \\ a_{21} & b_2 & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & b_n & a_{n3} & \cdots & a_{nn} \end{vmatrix}}{|A|}.$$

$$\ldots, \quad x_j = \frac{\begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & & & & & \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix}}{|A|}$$

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The rule is as follows: In the numerator of the quotient for x_j, replace the j^{th} column of A by the right-hand side column vector b.

Example

(1) Solve the system using Cramer's rule

$$2x - y - 3z = 1$$
$$x + 2y + z = 3$$
$$2x - 2y - 5z = 2.$$

Soln

Let A be the matrix of coefficient. Then

$$A = \begin{bmatrix} 2 & -1 & -3 \\ 1 & 2 & 1 \\ 2 & -2 & -5 \end{bmatrix}, \quad \text{Thus } |A| = \begin{vmatrix} 2 & -1 & -3 \\ 1 & 2 & 1 \\ 2 & -2 & -5 \end{vmatrix} = -5 \neq$$

Replacing the first column by the vector $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

$$x = -\frac{1}{5} \begin{vmatrix} 1 & -1 & 3 \\ 3 & 2 & 1 \\ 2 & -2 & -5 \end{vmatrix} = -\frac{5}{5} = -1$$

Replace the Second column by the vector $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

$$y = -\frac{1}{5} \begin{vmatrix} 2 & 1 & -3 \\ 1 & 3 & 1 \\ 2 & 2 & -5 \end{vmatrix} = \frac{-15}{-5} = 3.$$

$$z = -\frac{1}{5} \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 2 & -2 & 2 \end{vmatrix} = -\frac{10}{5} = -2.$$

$x = -1$, $y = 3$, $z = -2$

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EXERCISE

Use cramers rule to solve

$$2x - 3y + 2z = 9$$
$$3x + 2y - z = 4$$
$$x - 4y + 2z = 6$$

Echelon Form And Reduced-Row Echelon Form

Def: A Matrix is in echelon form if it meets the following conditions.

1. All nonzero rows are above any rows of all zeros
2. Each leading entry of a row is in column to the right of the leading entry of the row above it
3. All entries in a column below a leading entry is zero.

Def: Reduced Row-echelon form :- a matrix A said to be a reduced echelon form if it satisfies all the following

1. All nonzero rows are above any rows of all zeros.
2. A leading entry in any nonzero is 1
3. Each column that contains a leading 1 has zeros everywhere else.

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4) In any two successive nonzero rows, the leading 1 in the lower row occurs further to the right than the leading 1 in the higher row.

Def:- Augmented Matrix:- Suppose a system has m equations in n variables, as defined in ① and. Then the augmented matrix of the system is the $[m \times (n+1)]$ matrix whose first n columns are the column of A and whose last column $(n+1)$ is the column vector b, written as $[A|b]$. In other words, given the equations $Ax = b$, if write the elements of b within the matrix A, we obtain the augmented matrix B of the given set of equations $Ax = b$ written as $[A|b]$

Example

use the reduced echelon form to solve

$$2x_1 - 3x_2 + 2x_3 = 9$$
$$3x_1 + 2x_2 - x_3 = 4$$
$$x_1 - 4x_2 + 2x_3 = 6.$$

Soln
The matrix form of the system is $Ax = b$, and the augmented matrix is of the form $[A|b]$
Thus we have the augmented matrix as $\left[\begin{array}{ccc|c} 2 & -3 & 2 & 9 \\ 3 & 2 & -1 & 4 \\ 1 & -4 & 2 & 6 \end{array}\right]$

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(14) (47)

EXERCISE

Use the reduced row echelon form to solve the system of equations below

$$x + 2y + 3z = 9$$
$$2x - y + z = 8$$
$$3x \quad - z = 3$$

Ans: $x = 2,\ y = -1\ z$

Gaussian - Elimination Method

Example:-

Use the Gaussian elimination method to sol the system of equation

$$2x - y - 3z = 1$$
$$x + 2y + z = 3$$
$$2x - 2y - 5z = 2.$$

Solution

The matrix form of the system is

$$\begin{bmatrix} 2 & -1 & -3 \\ 1 & 2 & 1 \\ 2 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

We shall reduce the augmented matrix to upp triangular form.

$$\left[\begin{array}{ccc|c} 2 & -1 & -3 & 1 \\ 1 & 2 & 1 & 3 \\ 2 & -2 & -5 & 2 \end{array}\right] \quad R_1 \sim R_2.$$

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Now using the elementary row operations on the augmented matrix we have.

$$\begin{bmatrix} 2 & -3 & 2 & 9 \\ 3 & 2 & -1 & 4 \\ 1 & -4 & 2 & 6 \end{bmatrix} \quad R_3 \sim R_1 \quad \begin{vmatrix} 1 & -4 & 2 & 6 \\ 3 & 2 & -1 & 4 \\ 2 & -3 & 2 & 9 \end{vmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & -4 & 2 & 6 \\ 0 & 14 & -7 & -14 \\ 0 & 5 & -2 & -3 \end{bmatrix} \quad R_2 \to \frac{R_2}{14} \quad \begin{bmatrix} 1 & -4 & 2 & 6 \\ 0 & 1 & -\frac{1}{2} & -1 \\ 0 & 5 & -2 & -3 \end{bmatrix} \begin{matrix} R_1 \to R_1 + 4R_2 \\ \\ R_3 \to R_3 - 5R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & -\frac{1}{2} & -1 \\ 0 & 0 & \frac{1}{2} & 2 \end{bmatrix} \quad R_2 \to R_2 + R_3 \quad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 2 \end{bmatrix} \quad R_3 \to 2R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

This is the reduced echelon form of the augmented matrix. Therefore the matrix equation becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

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Hence $x_1 = 2$, $x_2 = 1$ and $x_3 = 4$.